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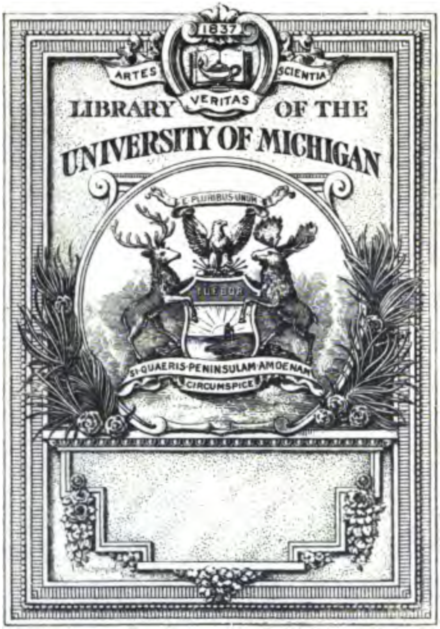
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ELEMENTARY DYNAMICS.

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WITH NUMEROUS EXAMPLES.

BY

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CALCUTTA.

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PREFACE.

THE following elementary treatise on Dynamics (commonly called Mechanics) will be found to differ in some important respects from the text-books in common use. Some of these differences will be here noticed. The term *Dynamics* is used in its true logical sense as the science which treats of the action of force on matter whether it maintains relative rest or produces acceleration of relative motion, and Dynamics is thus sub-divided into *Statics* and *Kinetics*. In both of these branches the unit of force adopted is the *Kinetic unit*, derived from the unit of mass which is taken to be the pound avoirdupois. As a secondary unit of force, convenient in certain cases and sufficiently accurate for most practical purposes, although varying in value at different places, the common *Gravitation unit*, the weight of the unit of mass (that is, the weight of one pound), has been adopted.

It is presumed that, at the present day, there can be no question as to the great advantages, both as regards simplicity and clearness for beginners, and also as regards scientific accuracy, of Gauss's method over the methods which have been in common use, in which the so-called Statical unit of force is a variable quantity, and in which it is necessary in the Kinetic branch to employ a different unit of force, the so-called Dynamical unit, from which a variable and very inconvenient unit of mass is derived, altogether different from the fixed standards of mass (commonly, but incorrectly, called standards of weight) used in commerce.

Care has been taken throughout the book, both in the text and in the examples, to avoid the use of the term *weight*, except in its true sense as the apparent force of gravity on a body.

The chapters on Kinematics, which are, of course, independent of the rest of the book, are placed in the order

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in which the subjects are now usually read. Kinematics, however, involving only the ideas of space and time, is a much less complex subject than Dynamics, which introduces the further ideas of force and mass, and might with advantage be studied by beginners before even commencing the Static branch. The chapter on Dynamical Laws and part of the chapter on the Force of Gravity might also be read as an introduction to the whole subject.

The importance, however, particularly for beginners, of clearly separating results and formulas derived from conventions, with regard to methods of estimating velocity and acceleration of velocity, from conclusions depending on the laws which are observed to regulate the action of force on matter, cannot, it is thought, be too strongly insisted upon. Nothing, perhaps, in Dynamics has given rise to more confusion of ideas with beginners than the use of the expression "accelerating force" for *acceleration of velocity*, and the arbitrary distinction drawn between "accelerating" and "moving" forces.

Any treatise on Dynamics, in the present state of science, would be incomplete without some reference to the Principle of Conservation of Energy. It has been attempted in chapter VIII of the Statics to explain the scientific sense in which the terms *work* and *energy* are now used, and to illustrate the Law of Energy, and some consequences from it, in connection with the Simple Machines. Further explanations and illustrations are given in the chapter on Energy in Kinetics, and it is hoped that these chapters combined may form a sufficient introduction to the subject for those readers who are commencing the study of Physical Science.

This little treatise was originally designed to supply a text-book on Dynamics, in conformity with late methods and improvements, for the pass students of the Calcutta University, and the greater portion of it has been compiled from notes, taken from various sources, which the author has used for some years with his classes. The book, however, as it now stands, contains most of the propositions which can be advantageously investigated by the aid of Elementary Algebra, Geometry and Trigonometry, such as are usually found in elementary treatises, together with some additions. Of the examples, very few are original; some have been taken from University Examination papers, and others, which have been considered as more or less common property, have been taken from various sources, including some of the text-books in common use. In many of these examples the wording and notation have been altered and a few have been altogether re-cast.

It seems scarcely necessary in a text-book, which, from its very nature and the purpose for which it is intended, must be, more or less, a compilation of well known methods and propositions, to specify, in detail, sources from which information has been obtained. It should, however, be stated that, as regards arrangement, method and demonstrations, much has been derived from Sir W. Thomson and Professor Tait's Treatise on Natural Philosophy. The thanks of the author are due to Mr. M. H. L. Beebee for many valuable suggestions in the chapters on Kinematics and Kinetics.

It is felt that some apology may be due for imperfections in type and diagrams, which will be noticed throughout the book. Any one who has had the misfortune to be obliged to get Mathematical printing done, and diagrams engraved, in this country, will fully appreciate the difficulties which had to be encountered in attempting to secure even tolerable accuracy and neatness of arrangement. The scarcity of mathematical type rendered it necessary that each form should be passed and printed off before the next was commenced, and thus mistakes and inaccuracies which might have been corrected, had a general revision been possible, may have escaped notice. It is hoped, however, that no errors of a serious nature have been overlooked. The publishers, with the limited means at their disposal, have done all that it was possible for them to do.

CALCUTTA: }
January, 1874. }

W. G. WILLSON.

ERRATA.

Page 13, example 7, for “32·188” read “32·088.”

Page 21, line 10, for “$\alpha - \beta$” read “$\alpha + \beta$.”

Page 37, line 10, after “outwards” insert “at the middle points of the sides of the triangle.”

Page 41, example 31, for “30 yards” read “60 yards.”

Page 160, last line, for “30·112” read “32·112.”

Page 171, example 6, read “*Ans.* 26·9.”

Page 203, last line but one, for “r” read “r^2.”

Page 209, line 18, for “C” read “P.”

Page 228, lines 5 and 6 from the bottom, for “$\alpha\,\Sigma m$” read “$\bar{\alpha}\,\Sigma m$.”

STATICS.

CHAPTER I.

INTRODUCTION.

1. **Matter.**—Matter is anything which can be perceived by the senses. A portion of matter limited in every direction, and therefore of determinate figure and volume, is called a body.

2. **Material Particle.**—A material particle is a portion of matter so small that, as far as its *position* is concerned, it may be regarded as a mathematical point; for shortness sake we shall speak of it as a particle.

3. **Motion and Rest.**—A body is in motion when its parts occupy successively different positions in space. When this is not the case, the body is at rest. We can only distinguish the state of rest or motion of a body by comparing it with other bodies; and for this reason all motions which come under our observation are *relative.* We thus speak of the state of rest or motion of a body relatively to the bodies which surround it.

4. **Force.**—That which produces, or tends to produce, a change in a body's state of relative rest or motion is known by the general term force. Matter may also be defined to be that which can be acted upon by, or can exert, force. Force, and by it matter, is the direct object of all our senses; more obviously, force is the object of our muscular sense.

5. **Dynamics.**—Dynamics is the science which treats of the action of force on matter, whether it maintains relative rest or produces relative motion. It is divided into *Statics* and *Kinetics.* Statics treats of the action of the forces which

B

keep a body at rest. Kinetics treats of the action of forces which produce motion in a body.

Kinematics, which may be considered as introductory to Kinetics, investigates the possible motion of a body, or system of bodies, *without* any reference to the cause by which the motion might be produced. It is the purely geometrical science of motion in the abstract, and is sometimes called "Geometry of pure motion."

6. EQUILIBRIUM OF FORCES. — Several forces acting simultaneously on a body, or particle, may so modify each other's effects, that no motion of the body or particle can take place; the body or particle is then said to be in equilibrium; or, the forces are said to maintain equilibrium. A system of forces is said to be in equilibrium when a body would remain at rest if it were acted upon by that system alone.

7. EQUAL FORCES.—*Two forces are equal when they are such that if they were applied to a particle in opposite directions they would maintain equilibrium.*

If two *equal* forces act on a particle in the same line, and tend to produce motion in the same direction in the line, they are equivalent to a single force double of either; if three equal forces be thus united, we obtain a triple force, and so on. When we speak of a force acting on a particle being a multiple of another force, we mean that the first may be supposed to be made up of a certain number of forces each equal to the second and acting in the same direction.

In this way we can measure forces by comparing their effects with that of some known force, taking the known force as the unit to which all other forces are to be referred. Thus by a force P we mean a force equal to P times the unit force, or P units of force; P being the abstract number which expresses the ratio of the force P to the unit force.

8. FORCE OF GRAVITY.—The force with which we are most familiar is the force of gravity, or the force of the earth's attraction. Experience teaches us that all bodies, when allowed to fall freely, do so in vertical directions, that is in directions perpendicular to the surface of a liquid at rest,

If a body be prevented from falling by the interposition of some other body, such as a table or the hand, there is a certain pressure exerted on the table or the hand; it will be shown in Kinetics that this pressure is caused by the earth's attraction on the body, it is called *the force with which the body gravitates towards the earth.* It will also be explained that this *gravitating force,* for the same body, is different when the body is taken to different places on the earth.

Obs.—The student will understand hereafter that the pressure which a body exerts against the earth is the resultant of the true force of gravity, and the so-called centrifugal force on the body. This has been called the *apparent* force of gravity on the body, and, except the contrary is expressed, it is in this sense that "force of gravity" is to be taken.

9. MASS.—The quantity of matter which a given body contains is called its mass.

Two bodies contain the same mass, or the same quantity of matter, when they gravitate towards the earth at the *same place* with the same force. The mass of any body is proportional to the force with which it gravitates.

A more general definition of the equality of two masses, but which includes that given here, will be explained in the chapter on Dynamical Laws. Masses of bodies are commonly measured, according to this definition, by a machine called a common balance. The use of this machine, in everyday life, in measuring quantities of matter, is more or less familiar to every one; it will be described in the chapter on Machines. The most convenient unit of mass for scientific investigation is the unit used for practical purposes.

The British unit of mass is the mass of an imperial pound avoirdupois.

In India and in France the unit of mass is the mass called a Sér or Kilogramme, which is about equivalent to two pounds.

Obs.—Units of mass are commonly called units of *weight,* and weight is often used in different senses. It sometimes means the mass of a certain body; at other times, the force with which the body gravitates. This double meaning of weight has led to considerable confusion and difficulties in elementary treatises on Dynamics and Hydrostatics. In this treatise, the word *weight*, whenever used, is to be considered as synonymous with *gravitating force.*

10. KINETIC UNIT FORCE.—In choosing a unit by which to measure force, one should be selected which shall be independent of locality. The force with which a given body gravitates is different at different places on the earth, and on this account cannot furnish a unit force independent of locality. An absolute unit of force has only been obtained from kinetic principles.

The Kinetic unit force is the force which, acting on a national standard unit of matter during the unit of time, generates the unit of velocity.

The British unit force is the force which, acting on an avoirdupois pound of matter for a second, generates a velocity of a foot per second.

This is known as the "Kinetic unit force," or "Gauss's absolute unit;" it will be explained in the chapter on Dynamical Laws. It has been thought better to define it formally here, but it was not necessary to do so. It will be sufficient for the student to know its value compared with the force with which a pound avoirdupois gravitates at a certain locality.

A pound at the equator gravitates with a force 32.088 times the Kinetic unit force. Hence the Kinetic unit force is $\frac{1}{32.088}$ part of the force with which a pound gravitates at the equator, or roughly speaking about the force with which a mass of half an ounce gravitates.

Let g be the gravitating force, or weight, of a mass of one pound (the unit of mass), expressed in Kinetic units, for any locality; then, since the weight of a body is proportional to its mass, Art. 9, the weight W, in that locality, of a body whose mass is m pounds (or, of m units of mass) is given b

$$W=mg.$$

The weight of the unit of mass is different in different localities; thus, at Edinburgh $g=32.2$, and g varies very slightly from this value throughout Great Britain and Ireland. At Calcutta $g = 32.112$.

The values of g vary with the latitude of the place, from $g = 32.088$ at the equator to $g = 32.2527$ at the poles. The extreme values of the weight of the unit of mass thus differ by a quantity which is the $\frac{1}{194}$th part of the least value. Thus a body which gravitates with a force of 194 Kinetic units at the Equator would, if it could be removed to the Pole, gravitate there with a force of 195 Kinetic units.

11. Gravitation Unit Force.—The weight of the same body is different when the body is taken to different places on the earth. But from what has been said it will be seen that the difference is small. For most practical purposes the difference may be neglected, but when strictly accurate scientific measurements have to be made, such as the measurements of Electrical and Magnetic forces, it must be considered. The force of gravity furnishes a simple and convenient unit of force sufficiently accurate for most practical purposes.

The gravitation unit force is a force equal to the force with which one pound of matter gravitates, and is called the gravitating force of one pound, or shortly, a force of one pound.

Thus the gravitating force, or weight, of a body whose mass is m pounds, expressed in gravitation units, is a force of m pounds; when expressed in Kinetic units, it is mg Kinetic units, g having the particular value for the locality in question.

12. Force Estimated Numerically.—A force may be estimated numerically by the weight of the body which it is capable of counterpoising; for example, if a body whose mass is m pounds be supported by suspending it freely by a string, the force exerted vertically upwards by means of the string is, in gravitation units, a force of m pounds; in Kinetic units, a force mg.

13. Specification of a Force.—There are four elements which completely determine a force: (1) its point of application, (2) the line in which it tends to produce

motion, (3) the direction along the line to which the motion tends, (4) its magnitude or intensity.

When a force acts on a material particle, we may, according to the abstract conception of such a particle, Art. 2, consider the force as acting at a point. In reality, however, the place of application of a force is always either a surface or a portion of space occupied by matter. The point of a needle, or the sharpest edge, is still a surface, and acts as such to the bodies to which it may be applied. As an example of the place of application being a portion of space occupied by matter, we may take the force of gravity, whose place of application is the whole matter of the body which we consider. Thus there are two kinds of force distinguishable by the place of their application,—force whose place of application is a solid, and force whose place of application is a surface. Thus when a heavy body rests on a table, force of the first kind acting downwards is balanced by force of the second kind acting upwards.

The second element in the specification of a force is the line in which it tends to produce motion. If the place of application of a force be regarded as a point, a line through that point drawn towards the direction of tendency to motion is called the *line of action* of the force. In the case of force whose place of application is a surface, or a solid, we shall find that it is frequently possible to find a line and a point in that line, such that a single force acting at the point and along the line would produce the same effect as is actually produced.

14. REPRESENTATION OF FORCES BY LINES. — It is evident from what has been said that three of the elements which determine a force can be immediately represented geometrically. The place of application of a force may be represented by a point. The line of action of the force may be represented by a straight line through the point; an arrow-head on the line will indicate the direction towards which the force acts in the line. But we may also make use of a portion of this line as a geometrical representation of the intensity of the force. Measure off from the point, along the line, a portion containing as many units of length

as the force to be measured contains units of force: this portion of the line will conveniently represent the relative magnitude of the force. A line may thus be made to represent three of the elements of a force absolutely, and the fourth relatively.

For example, suppose two forces to act on a particle whose position is indicated by the point O. Let one of them be equal in magnitude to P units of force; the other to Q units of force.

From O draw OA in the direction of the force P, and let OA contain as many units of length as the force P contains units of force.

Similarly, measure off OB in the direction of Q, and containing as many units of length as Q contains units of force.

Then OA and OB represent the forces P and Q, absolutely as regards their point of application and directions, and relatively as regards their magnitudes.

The abstract numbers which denote the lengths OA and OB are the same as the abstract numbers which denote the forces P and Q. Thus—

$$\frac{OA \text{ units of length}}{OB \text{ units of length}} = \frac{P \text{ units of force}}{Q \text{ units of force}}.$$

This suggests another method of procedure. Take on OA any length OA, and take OB, so that $P : Q = OA : OB$, then the ratio of the forces is completely determined by the the ratio of the lengths, and if the absolute magnitude of one force is known the other is determined.

Thus, to represent the forces P and Q, it is sufficient to draw lines in the proper directions from O, and to take on these lines *any* two portions, measured from O, *proportional* to the magnitudes of the forces.

15. A RIGID BODY.—A solid body may be assumed to be composed of an aggregation of material particles; since experiments show that any body is divisible into successively smaller and smaller portions, if sufficient force be used to overcome the mutual actions of the parts of the body. For purposes of reasoning it is convenient to form

an abstract conception of a body perfectly hard and perfectly strong. Such a body is called rigid, and may be defined as a *body in which the relative positions of the particles cannot be altered by the application of any force however great.* No body in nature is perfectly rigid; every body yields more or less to the forces which act on it. If, however, the forces applied to a body are not sufficiently powerful to alter, to a sensible extent, the relative positions of its particles, it may be considered rigid, and this is, in many cases, approximaely true for solid bodies. As in this work rigid bodies mostly are considered, we shall often, in future, use the word *body* in the sense of *rigid body.*

16. The reasoning about rigid bodies is extended to other bodies by the following axiom—

If any forces, acting on a body of any kind, produce equilibrium, we may suppose any portions of the body to become fixed or rigid, or rigid and fixed, without destroying the equilibrium.

17. MOVING AND RESISTING FORCES.—There are two kinds of forces which may be distinguished by the mode of their action: (1) forces which tend to produce motion; (2) forces which tend to resist or change motion. To the latter belong all kinds of resistance, reaction, and friction. They have no tendency to produce motion, but have a tendency to change and modify it.

By motion is here to be understood molar or sensible motion. The student of Physical Science is aware that these resisting forces do produce molecular motion in the forms of heat, electricity, &c.

Resisting forces are determined in amount and direction by the moving forces which act against them. The principle on which this determination depends is expressed in Newton's third law of motion:

To every action there is always an equal and contrary reaction; or the mutual actions of any two bodies are always equal and oppositely directed.

This law is generally considered self-evident as far as it is used in Statics, and in most books is not formally stated.

As a simple example of a resisting force, suppose a gravitating particle to rest on a horizontal table. The gravitating force of the particle is a moving force which would cause the particle to fall in a vertical direction, if the table were removed. This moving force, therefore, acts vertically downwards. Again, the table supports the particle; and according to Newton's law, the resisting force, or the reaction of the table, is exactly equal to the gravitating force of the particle, and its direction is vertically upwards. The reaction of the table is in this case the resisting force, and its amount and direction are determined by the amount and direction of the moving force acting against it.

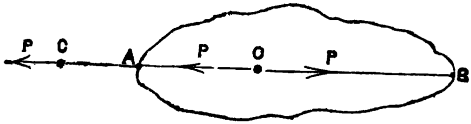
The terms *pressure*, *tension*, *reaction*, &c., are often used as synonymous with *force*.

18. SUPERPOSITION OF FORCES.—*The effect of any system of forces will not be altered by the addition or subtraction of another system of forces in equilibrium.*

Hence, if two systems of forces, when acting separately, would keep a body in equilibrium, they would also keep the body in equilibrium when acting on it simultaneously.

The principle of this article may be regarded as a physical axiom.

19. TRANSMISSION OF FORCE BY A RIGID BODY.—Suppose a force P to act at the extremity A of a rigid rod in the direction of its length.



Let the other extremity, B, of the rod be fixed. We may suppose the rod to be a part of a rigid body AB.

The force P acting at A is transmitted by the rigid body to any point, O, on its line of action, AB, with undiminished effect. Hence P may be supposed to act at any point, O, on its line of action.

This is generally regarded as axiomatic or as a result of experience, and is called the principle of *transmissibility of force*. The principle may be thus stated—

When a force acts on a rigid body, its effect will be unchanged at whatever point of its line of action it be supposed to act, provided the point be either one of the points of the body or be supposed to be rigidly connected with it, the direction of the force remaining the same.

20. As this principle is the foundation of Statics, and usually presents considerable difficulties to beginners, we give another illustration of the amount of assumption involved in it.

At O any point in the line AB, suppose two forces, each equal to P, applied in opposite directions along the line AB. The introduction of these forces in equilibrium will not affect the action of P. Art. 18.

If we assume that P acting at O along OB equilibrates P acting at A along AC, an easy extension of Art. 7, these may be removed, Art. 18, and there remains P acting at O along OA. Thus the effect of P acting at A is the same as the effect of P acting at O.

21. It is easily seen that the principle is a consequence of the nature of a rigid body and the law of action and reaction. Art. 17.

The particle of the body at A is drawn by the force P; this particle must then be kept in equilibrium by a force equal to P in the direction AB, Art. 7, which is the action on it of the next particle along the line AB. The first particle reacts on the second with an equal and opposite force. This system of actions and reactions takes place along the whole line AB, and by this means the force P is transmitted along the line with undiminished effect. This is sometimes expressed by saying that the tension or strain along the rod AB is the same throughout its length, and is measured by the force P.

We have supposed P to act from A towards C, that is to pull the rod so as to tend to extend it, but the same conclusions would follow if we supposed P to act at A from C towards A, that is to push the rod so as to tend to compress it. Since the rod is supposed to be rigid, it is, of course, inextensible and incompressible.

22. TRANSMISSION OF FORCE BY A FLEXIBLE INEXTENSIBLE STRING.—Let AB, Fig. Art. 19, be a flexible, but inextensible, string attached to a fixed point B, and stretched by a force P acting from A along its length.

Since the string is in equilibrium, we may suppose it to be rigid without destroying the equilibrium, Art. 16; in fact, since the string is inextensible, it resists the force P as if it were rigid, and, as in the last article, the force P is equally transmitted to any point of the string. This is expressed by saying that—

The tenison of a string stretched by a force at one end is the same throughout its length, and is measured by the force which stretches it.

The force must act in such a direction as to stretch the string, that is, must be a pull, and not a push. In this a string differs from a rod in its power of transmitting force, for while a rod will transmit equally force which tends to extend it and force which tends to compress it, a string can only transmit force which tends to extend it.

But a string, by reason of the flexibility of its parts, possesses the important property of being able to transmit a force, which tends to extend it, in any direction with undiminished intensity. This is effected by turning the string to the required direction by making it pass over smooth surfaces.

The tension of the same string, which is in contact with smooth surfaces only, is the same throughout.

This will be more fully explained hereafter; for the present we may assume it as an axiom, or as a result of experience.

23. SMOOTH SURFACE.—By a smooth surface is meant a surface which opposes no resistance whatever to the motion of a body upon it. A surface which does resist the motion of a body along it is said to be rough, and the resisting force is called the *force of friction*. A smooth surface can resist pressure only in the direction perpendicular to the surface at the point where the pressure is applied, or *the direction of the reaction of a smooth surface*

is the normal to the surface at the point. This may be regarded as axiomatic, but we will hereafter show that it is a consequence from the definition of smoothness. When the contrary is not expressed, we always suppose the surfaces of bodies to be smooth. Smoothness is another abstraction introduced for the sake of simplifying the problem of equilibrium. There is no such thing in nature as a smooth surface; but we will see further on how friction modifies the problem.

24. REMARKS.—Many of the axioms and definitions in this chapter will probably at first present considerable difficulties to the student. "An axiom is a proposition the truth of which must be admitted as soon as the terms in which it is expressed are clearly understood." But the student must bear in mind that "physical axioms are axiomatic to those only who have a sufficient knowledge of the action of physical causes to enable them to see at once their necessary truth." The first principles of any physical science are necessarily general and abstract, and therefore cannot be thoroughly comprehended until the subject itself has been read and considered practically, and the meanings of the axioms traced in the applications made of them. The student will find that the more familiar he becomes with the idea of force and the mode of its action, the clearer will the meaning and truth of these first principles become in his mind.

In theoretical or abstract dynamics, we reason about material particles, rigid bodies, and ideal properties of bodies, just as in geometry we reason about points, lines, and surfaces, which are abstractions and have no existence in nature. The solutions of problems thus obtained are first approximations to the solutions of the actual problems, and in practical applications are either near enough to the reality, or further approximations can be made by means of a more refined analysis.

EXAMPLES.

1. If a force of 10lbs. be represented by a straight line 6 inches long, what force will be represented by a line 2 feet long? *Ans.* 40lbs.

2. A force of 25 units is represented by a line 5 inches long; what length would represent a force of 35 units? *Ans.* 7 inches.

3. If a force of P units be represented by a line a inches long, what force will be represented by a line b inches long?

Ans. $\frac{b}{a}P$.

4. A line a inches long represents a force of P lbs.; what is the length of the line which would represent a force of Q lbs.?

Ans. $\frac{Q}{P}a$.

5. If a force of 100lbs. be represented by the diagonal of a square, what force would be represented by a side.

Ans. 70. 71lbs.

6. A string hanging vertically from a fixed point supports a mass of 4lbs. at its extremity, and a mass of 6lbs. at its middle point; find the tensions of the two parts of the string.

Ans. Tension of upper portion = 10lbs.
" lower " = 4lbs.

7. What force, estimated in Kinetic units, would support at the Equator a body whose mass is one ton? Here $m = 2240$; $g = 32.088$. *Ans.* 71877.12.

8. What force would support the same body at the Poles of the Earth? Here $g = 32.2527$. *Ans.* 72246.048.

9. Find in Kinetic units the difference in weight of a mass of one ton at the Poles and the Equator? *Ans.* 368.928.

1 ton = 2240

10. The weight in Kinetic units of a body is W, at a place where the weight of the unit of mass is g; find the change in weight of the same body at a place where the weight of unit of mass is g'.

Ans. $W\frac{g'-g}{g}$

CHAPTER II.

FORCES MEETING AT A POINT.

25. A system of forces acting on a particle may, according to the definition of a particle, be considered as a system of forces meeting at a point.

When a system of forces, whose directions meet at a point, acts on a rigid body, we may, by Art. 19, suppose all the forces to act at that point; and it is evident that if they maintain equilibrium in the body, they would also maintain equilibrium, if they acted, instead, on a particle; and *vice versâ*.

When a number of forces act on a particle, or when they act on a body in such a manner that their directions meet at a point, we speak of them as "forces meeting at a point."

They are in equilibrium when they would maintain equilibrium either in a particle or in a rigid body.

26. RESULTANT AND COMPONENTS.—*The resultant of a system of forces meeting at a point is the single force whose effect is equal to the combined effect of all the forces of the system.*

The forces of the system are called *components* with reference to this resultant. When the system is in equilibrium, the resultant is zero.

27. *Every system of forces not in equilibrium, whose directions meet at a point, has got a single resultant.*

We may suppose the forces to act on a particle at the point where they meet. Suppose then the forces to be simultaneously applied to this particle; it is clear that the particle would begin to move in some definite direction.

It is evident that a single force could be found of such a magnitude that, if it acted on the particle in a direction opposite to that in which motion would take place, it would prevent the motion. This force would then be equal in magnitude and opposite in direction to the combined effect of all the forces of the system.

The forces of the system have therefore got a single resultant equal in magnitude to this force and acting in the direction in which motion would take place.

28. *If any number of forces meeting at a point be in equilibrium, any one of them is equal in magnitude and opposite in direction to the resultant of all the rest.*

For we may suppose the forces to act on a particle at their point of intersection. If any one of the forces were removed, it is evident that the particle would begin to move in the direction of the force removed. This force therefore counteracts the combined effect of all the rest, and is therefore equal and opposite to their resultant.

29. COMPOSITION AND RESOLUTION. — The process by which a system of forces is replaced by one or more forces, equivalent in effect to the system, is called the *composition of forces*. When a single force is replaced by others equivalent in effect, the process is called *resolution*.

Before determining the conditions of equilibrium of systems of forces, it will be necessary to deduce rules for the composition of forces, that is, for finding their resultants. The conditions of equilibrium will then express the relations which must exist among the forces in order that their resultants may be zero.

30. *To find the resultant of any number of forces acting in the same straight line, and the condition of equilibrium.*

It is evident, by Art. 7, that the forces which act in the same direction along the line may be replaced by a single force equal to their sum. The forces which act in the opposite direction to this force may also be replaced by a single force equal to their sum. It is clear that these two forces, acting in opposite directions, may be replaced by a single force equal to their difference and acting in the direction of the greater.

Hence the resultant of all the forces is equal to their algebraic sum, those in one direction being considered positive, and those in the opposite direction negative.

The condition of equilibrium is that, the algebraic sum of the forces should be equal to zero.

31. *To find the line of action of the resultant of two equal forces, not in the same straight line, acting on a particle, or meeting at a point.*

It is obvious that the resultant must lie in the plane of the two forces. For no reason can be assigned in favor of its lying on one side of this plane which could not, with equal validity, be assigned in favor of its lying in a similar position on the other side, and we know, by Art. 27, that there can be only one resultant, therefore it must lie in the plane of the forces.

Moreover, it is evident that the resultant must *bisect* the angle between the components. For since the component forces are equal, there is no reason why the resultant should make with one of the forces an angle different from that which it makes with the other.

If we measure from the point where the forces meet, along their lines of action, two equal portions to represent them, Art. 14, and construct the parallelogram having these two lines for adjacent sides, the diagonal of this parallelogram will be equally inclined to its adjacent sides, since they are equal. The direction of the resultant is therefore represented by the diagonal of the parallelogram drawn from the point where the forces meet.

32. PARALLELOGRAM OF FORCES.—*If two forces acting at a point be represented in magnitude and direction by two straight lines drawn from that point, and if a parallelogram be constructed having these two lines for adjacent sides, then the diagonal of the parallelogram which passes through the point of application of the forces will represent their resultant in magnitude and direction.*

Duchayla's proof of this important theorem, called the *parallelogram of forces*, is based on the following:—

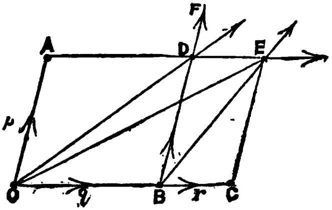
(1.) The direction of the resultant of two equal forces meeting in a point is the diagonal of the parallelogram, drawn from the point, of which the components are adjacent sides. Art. 31.

(2.) The point of application of a force may be supposed to be transferred to any point in the line in which the force acts without altering the effect of the force, the points being conceived to be rigidly connected. Art. 19.

The proof is divided into three parts: in the first part, the direction of the resultant is deduced, by the method of mathematical induction, in general, from the particular case of two equal forces, for which it is already known, the forces being commensurable; in the second part, this proof is extended to incommensurable forces; in the third part, the magnitude of the resultant is deduced from the knowledge of its direction.

33. *The line of action of the resultant of any two commensurable forces meeting at a point is a diagonal of the parallelogram constructed on the lines which represent the forces.*

We shall first prove that if this proposition be true for any two forces, p and q, inclined at any angle, and also for two forces, p and r, inclined at the same angle, then it will be true for the forces p and $q+r$, inclined at the same angle.



Let the forces p and $q+r$ meet at O; take OA and OB to represent p and q in magnitude and situation, and BC to represent r in magnitude. Thus OC will represent $q+r$ in magnitude and situation, and r may be supposed to act at B instead of O without altering its effect. Art. 19. Complete the parallelograms $OADB$ and $BDEC$.

By our hypothesis OD is the line of action of the resultant of the forces p and q represented by OA and OB; let these forces be replaced by their resultant, and let it be supposed to act at D instead of O. Art. 19. Since two forces are in every way equivalent to their resultant, we may evidently replace the resultant at D by the original components parallel to their original directions, p acting along DF and q along DE; the former may be supposed to act at B, and the latter at E.

Again, p and r at B are represented by BD and BC, therefore, by hypothesis, their resultant acts along BE, and may be supposed to act at E. This resultant at E, and q at E, must have a single resultant passing through E.

Thus the effect of the forces, p and $q+r$, acting at O, is the same as that of a single force through E; therefore E is a point on their resultant, but O is also a point on their resultant, hence OE is the line of action of the resultant of p and $q+r$.

Thus, if the proposition be true for forces p and q and for p and r, it must be also true for forces p and $q+r$. But the proposition is true for equal forces p and p, Art. 31, therefore it holds for p and $2p$, and therefore for p and $3p$, and so on; hence it holds for p and mp, where m is any integer.

Since it is true for (mp, p) and (mp, p) it holds for mp and $2p$, and therefore for mp and $3p$, and so on; hence it holds for mp and np, when m and n are both integers.

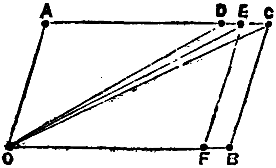
Thus the proposition is true for any two commensurable forces, $P = mp$ and $Q = np$.

34. *The line of action of the resultant of two incommensurable forces is also the diagonal of the parallelogram.*

This might be inferred from the fact, that although we cannot express the ratio of the incommensurable forces P and Q by any numbers exactly, yet we can always find two integers m and n (and therefore two commensurable numbers) whose ratio will represent the ratio of P to Q within any assigned degree of closeness.

It is usual to give the following indirect proof.

Let OA OB represent two incommensurable forces. Complete the parallelogram OA CB; and, if it be possible, let their resultant be in any direction, OD different from OC.



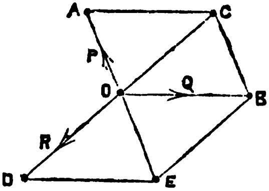
Divide OA into any number of equal parts, each less than DC; measure off from A along AC parts equal to these, and let E be the extremity of the last part, which can thus be measured off; then E evidently lies between D and C. Draw EF parallel to OA, join OE.

The force represented by OB may be replaced at O by two forces, one represented by OF, the other by FB. Now the forces represented by OA and OF are commensurable, and therefore have a resultant in the direction OE. They may be replaced by this resultant, which, together with the force represented in magnitude by FB acting at O along OB, will give a resultant within the angle EOB, which lies nearer to OC than OD.

Therefore OD cannot be the direction of the resultant of the forces, and similarly it can be shown that no other line except OC can be the direction of their resultant.

The Parallelogram of Forces is thus demonstrated as far as the line of action of the resultant is concerned for any two forces whatsoever meeting in a point.

35. *Assuming that the diagonal represents the line of action of the resultant: to prove that it also represents the resultant in magnitude.*



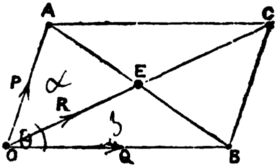
Let OA and OB be the lines of action of the forces P and Q. Take lengths OA and OB to represent these forces, and complete the parallelogram $OACB$; OC is the line of action of the resultant.

Let R be the unknown magnitude of the resultant of P and Q. Take OD in the opposite direction to OC to represent R. Then OD represents a force R equal in magnitude and opposite in direction to the resultant of P and Q. Hence the forces P, Q and R are in equilibrium. Art 28.

Complete the parallelogram $ODEB$. Then OE is the direction of the resultant of Q and R. But the direction of the resultant of Q and R must be opposite to P, since the forces are in equilibrium. Hence OA and OE are in the same straight line, and therefore $OEBC$ is a parallelogram.

Hence $EB = OC$; but $EB = OD$; therefore $OC = OD$. OC therefore represents the magnitude, as well as the line of action, of the resultant of P and Q.

The Parallelogram of Forces has now been completely demonstrated for any two forces meeting at a point.



36. Geometrical Constructions.—The following geometrical constructions for the resultant of two forces meeting at a point follow immediately from the Parallelogram of Forces.

(1.) Take any length OA to represent one of the forces P in magnitude and direction; from A draw AC parallel to the direction of Q, and so that $AC : AO = Q : P$; join OC; OC represents the resultant in magnitude and direction.

(2.) Take OA and OB to represent the forces acting at O in magnitude and direction; join AB, bisect it at E, and join OE. OE represents the direction of the resultant, and $2OE$ represents its magnitude.

37. The sides of the triangle OAC (or OBC) represent the magnitudes of the forces and their resultant; two of the angles are the inclinations of the resultant to the components, and the third angle is the supplement of the angle at which the components meet. The abstract numbers which represent the ratio of the forces to the unit force, or any multiples or submultiples of these numbers, may be taken as the numerical values of the sides of the triangle. Of the six quantities P, Q, R and the angles at which they are inclined to each other, when any three (except the three angles) are given, the remaining quantities can be found by the rules of trigonometry. Some examples of these determinations will now be given.

38. *Two given forces* P *and* Q *meet at a given angle* θ*; required to determine the magnitude of the resultant* R, *and the angles* α *and* β *which it makes with the components.*

Let OA and OB (Fig. Art. 36) represent the forces P and Q; complete the parallelogram, then OC represents R.

In the triangle OAC we have by trigonometry

$$OC^2 = OA^2 + AC^2 - 2OA.\,AC \cos OAC$$

but $AC = OB,\ OAC = 180° - AOB = 180° - \theta$

$$\therefore OC^2 = OA^2 + OB^2 + 2OA.OB \cos\theta$$

but
$$\frac{OA}{P} = \frac{OB}{Q} = \frac{OC}{R}$$

$$\therefore R^2 = P^2 + Q^2 + 2PQ\cos\theta \quad \text{.........(1)}$$

Let the angle $AOC = \alpha$ and $BOC = \beta$,

we have
$$\frac{OA}{\sin\beta} = \frac{AC}{\sin\alpha} = \frac{OC}{\sin(180° - \theta)}$$

$$\therefore \frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{R}{\sin\theta} \quad \text{.........(2)}$$

From (1) R can be calculated, and then α or β from (2). When either α or β is determined, the other is found from the equation $\alpha + \beta = \theta$.

Cor. 1.—The angles α and β might also be determined by the equations—

$$\cos\alpha = \frac{P^2 + R^2 - Q^2}{2PR};\quad \cos\beta = \frac{Q^2 + R^2 - P^2}{2QR}.$$

Cor. 2.—The equations (2) express the fact that each of the forces P, Q, R is proportional to the sine of the angle between the other two. Again, since a force equal and opposite to R would equilibrate P and Q, and since the sines of supplemental angles are equal, *when three forces are in equilibrium at a point, each force is proportional to the sine of the angle between the other two.* On account of the importance of this proposition an independent proof will be given further on.

Cor. 3.—Equations (2) also express that *the resultant of two forces meeting at a point divides the angle between the forces into parts whose sines are in the inverse ratio of the forces.*

Cor. 4.—When three forces, P, Q, R, meeting at a point, are in equilibrium, any one of them is equal and opposite to the resultant of the other two. Hence if the angle between P and Q be denoted by (PQ) we have from (1)—

$$\cos(PQ) = \frac{R^2 - P^2 - Q^2}{2PQ};$$

$$\cos(QR) = \frac{P^2 - Q^2 - R^2}{2QR};$$

$$\cos(RP) = \frac{Q^2 - P^2 - R^2}{2PR}.$$

Obs.—When we speak of the ratio of a force to a line (or to the sine of an angle), the student must remember that the ratio is between two abstract numbers, one of which expresses the ratio of the magnitude of the force to the unit of force, and the other the ratio of the length of the line to the unit of length. Keeping this in mind, such expressions as

$$\frac{P}{OA}, \ \frac{P}{\sin\beta}$$

are intelligible, and are very convenient.

39. *Examples.*—(1.) The resultant of two given forces is greatest when they act in the same straight line in the same direction, and least when they act in the same straight line in opposite directions.

These results are evident geometrically, or may be inferred from formula (1) of Art. 38. For, in the first case, $\cos\theta = 1$, and therefore R is greatest and equal to $P + Q$. In the second case, $\cos\theta = -1$, hence R is least and equal to $P - Q$.

(2.) When the angle between the forces is a right angle $\sin\theta = 1$, hence—

$$R^2 = P + Q^2; \ \cos\alpha = \sin\beta = \frac{P}{R}.$$

These results are also evident geometrically by Euclid I, 47.

(3.) When the forces are equal, or $Q = P$,

$$R^2 = P^2 + P^2 + 2P^2\cos\theta = 2P^2(1 + \cos\theta);$$

but $\qquad 1 + \cos\theta = 2\cos^2\tfrac{1}{2}\theta;$

therefore $R^2 = 4\,P^2 \cos^2 \frac{1}{2}\,\theta;$

hence $R = 2\,P \cos \frac{1}{2}\,\theta;$

and $\alpha = \beta = \frac{1}{2}\,\theta.$

These results may also be inferred geometrically. For, in construction (2) of Art. 36, when $OA = OB$, OE bisects the angle AOB and is at right angles to AB—

hence $OE = OA \cos \frac{1}{2}\,\theta$

therefore $R = 2P \cos \frac{1}{2}\,\theta.$

40. *The conditions of equilibrium of three forces meeting at a point.*

It is evident, as in Art. 28, that in order that any one of the forces may counteract the effect of the other two, it must be equal in magnitude and opposite in direction to their resultant.

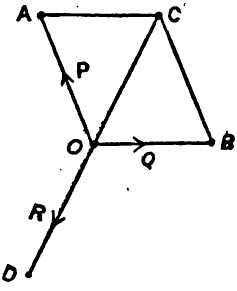
The conditions of equilibrium are therefore—

(1.) The directions of the three forces must lie in one plane.

(2.) Any one of the forces must be equal in magnitude and opposite in direction to the resultant of the other two.

The latter of these conditions may be expressed mathematically in various ways, some examples of which are given in the following articles.

41. *When three forces meeting at a point are in equilibrium; each force is proportional to the sine of the angle between the other two.*



Let the forces P, Q, R, acting in the directions OA, OB, OD, equilibrate.

Take any length OD to represent R: produce OD backwards, and take OC equal to OD, and complete the parallelogram by drawing CA and CB parallel to OB and OA.

Then OC, equal and opposite to OD, represents the resultant of P and Q; therefore, by the parallelogram of forces, OA represents P, and OB represents Q.

Let the angle $DOB = (QR)$, $AOB = (PQ)$ and $AOD = (PR)$.

Then—

$$\frac{OA}{\sin ACO} = \frac{AC}{\sin AOC} = \frac{OC}{\sin OAC};$$

therefore $$\frac{OA}{\sin COB} = \frac{OB}{\sin AOC} = \frac{OD}{\sin AOB};$$

hence $$\frac{P}{\sin (RQ)} = \frac{Q}{\sin (PR)} = \frac{R}{\sin (PQ)}.$$

42. *If three forces meeting at a point and lying in the same plane have magnitudes and directions such as to satisfy the relations*

$$\frac{P}{\sin (RQ)} = \frac{Q}{\sin (PR)} = \frac{R}{\sin (PQ)}$$

the forces either are in equilibrium, or one is the resultant of the other two.

Let the force P, Q, R act in the directions OA, OB, OD: produce OD backwards, and take any point in it C: from C draw CA and CB parallel to OB and OA, then—

$$\frac{OA}{\sin ACO} = \frac{AC}{\sin AOC} = \frac{OC}{\sin CAO};$$

or $$\frac{OA}{\sin COB} = \frac{OB}{\sin AOC} = \frac{OC}{\sin AOB};$$

hence $$\frac{OA}{\sin (RQ)} = \frac{OB}{\sin (QR)} = \frac{OC}{\sin (PQ)};$$

Combining this with the given relation we get

$$\frac{OA}{P} = \frac{OB}{Q} = \frac{OC}{R};$$

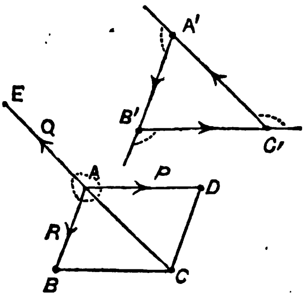
Therefore OA, OB, OC represent the forces P, Q, R in magnitude.

But, by the parallelogram of forces, OC represents the resultant of the forces P and Q, which are represented by OA and OB.

The resultant of P and Q is thus equal and opposite to R; hence the forces P, Q, R are in equilibrium.

Obs.—If the line of action of R, reckoned along its direction from O, lay within the angle AOB, it is evident, that if the above relation existed, R would be the resultant of P and Q. It will be easy to see from a figure, in any case, whether this is so or not.

43. TRIANGLE OF FORCES.—*If three forces meeting at a point be proportional and parallel to, and co-directional with, the sides of any triangle taken in order, they will be in equilibrium.*



Let the forces P, Q, R, meeting at the point A, be parallel and co-directional with the sides $B'C'$, $C'A'$, $A'B'$ of the triangle $A'B'C'$, taken in order as indicated by the arrow-heads.

Produce the direction of any of the forces Q, backwards to any point C, and from C draw CB and CD parallel to the directions of P and R.

The triangle ABC is parallel to the triangle $A'B'C'$, and is therefore similar to it, hence—

$$\frac{BC}{B'C'}=\frac{CA}{C'A'}=\frac{AB}{A'B'};$$

but by hypothesis—

we have $$\frac{P}{B'C'}=\frac{Q}{A'C'}=\frac{R}{A'B'};$$

hence $$\frac{P}{BC}=\frac{Q}{CA}=\frac{R}{AB}.$$

Therefore BC or AD represents P, CA represents Q, and AB represents R. But, by the parallelogram of forces, AC represents the resultant of the forces R and P, represented AB and AD. The resultant of R and P is equal and opposite to Q, therefore the forces are in equilibrium.

44. If we take two of the forces R and P parallel and co-directional with the sides $A'B'$ and $B'C'$, but the third force Q parallel and co-directional with $A'C'$, that is taking the side $A'C'$ in a different order from the other two, then Q, parallel to $A'C'$, will evidently be the resultant of R and P.

45. If the relation $\frac{P}{\sin(RQ)} = \frac{Q}{\sin(PR)} = \frac{R}{\sin(PQ)}$ exist, the forces will be proportional to the sides of any triangle drawn parallel to them, for the external angles of such a triangle are evidently equal to the angles between the forces, and the sines of the external angles are proportional to the opposite sides. This proposition is demonstrated independently in the next article.

46. CONVERSE OF TRIANGLE OF FORCES.—*If three forces meeting at a point be in equilibrium, and any three lines be drawn, in the same plane, parallel to the forces, and not meeting at a point, the sides of the triangle formed by the lines will be proportional to the forces.*

In the figure of Art. 43 let $A'B'$, $B'C'$, and $C'A'$ be any three lines parallel to the forces, R, Q and P, which are in equilibrium.

Take AD and AB proportional to P and R, and complete the parallelogram $ABCD$. Then AC represents the resultant of P and R, therefore AC and AE are in one straight line, and AC is proportional to Q.

The sides of the triangles ABC and $A'B'C'$ are therefore parallel. Hence the triangles are similar, and—

$$\frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{AB}{A'B'};$$

but
$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB};$$

therefore
$$\frac{P}{B'C'} = \frac{Q}{C'A'} = \frac{R}{A'B'}.$$

Which relation establishes the proposition.

47. The magnitudes of three forces meeting at a point, which could produce equilibrium, are subject to the condition

that the sum of any two cannot be greater than the third. Since, by Art. 46, the forces must be proportional to the sides of a triangle. When this condition is satisfied, the forces would always equilibrate, if properly directed.

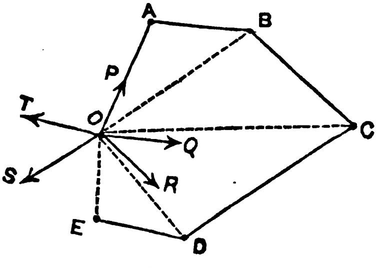
If one force be equal to the sum of the other two, there will be equilibrium when the forces act along the same straight line, the greatest in the opposite direction to the other two.

48. It is not necessary in Arts. 43 and 46 that the sides of the triangle $A'B'C'$ should be parallel to the forces. It will be sufficient if each of the sides $A'B'$, $B'C'$, $C'A'$ make the same angle with the directions of the forces R, P, Q; for in this case the triangles ABC and $A'B'C'$ will be similar as before. For example, the triangle $A'B'C'$ may be formed by lines perpendicular to the directions of the forces.

49. *To determine the resultant of any number of forces whose directions meet at a point.*

Take any two of the forces; find their resultant by the Parallelogram of Forces, and replace them by this resultant. Again, compound this resultant in a similar manner with a third, and replace them by their resultant; and so on. We finally arrive at a single force which is the resultant of the system.

50. *To determine by a geometrical construction the resultant of any number of forces whose directions meet at a point.*



Let the forces be P, Q, R, ... acting at the point O.

Draw OA in the direction of one of the forces P, and take on it any length OA.

From A draw AB parallel to the direction of Q, from B draw BC parallel to R, and so on; let the last force be T, and DE parallel to the direction of T; the lengths AB, BC .... DE being so taken that

$$\frac{OA}{P}=\frac{AB}{Q}=\ldots\ldots\ldots\ldots\ldots\ldots=\frac{DE}{T}$$

Join EO, then OE represents in magnitude and direction the resultant of all the forces.

For join OB, OC, OD .... Then, by Art. 36, OB represents in magnitude and direction the resultant of P and Q. We may replace them by their resultant represented by OB.

Again the resultant of this force and R is represented by OC, and finally the resultant of all the forces is represented in magnitude and direction by OE.

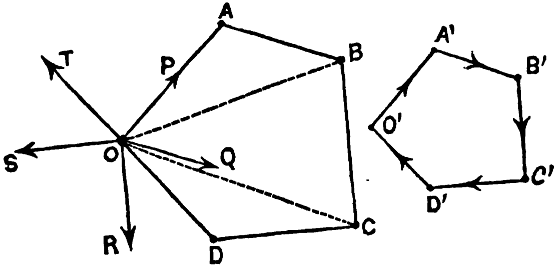
Cor.—It is evident that if the forces were in equilibrium, the last point arrived at, E, would coincide with O.

Obs.—In this construction it is not necessary that the forces should be in the same plane.

51.—Polygon of Forces.—*If any number of forces meeting at a point be proportional and parallel to, and co-directional with, the sides of a polygon taken in order, they will be in equilibrium.*

This might be deduced as a corollary of the last proposition. The following is an independent proof for a pentagon, and the same method applies in general.

Let the forces P, Q, R, ... acting at the point O, be parallel and proportional to the sides of polygon $O'A'B'C'D'$



From O draw OA in the direction of P, and on OA construct a polygon $OABCD$ similar and similarly situated to $O'A'B'C'D'$.

$$\frac{O'A'}{OA}=\frac{A'B'}{AB}=\dots\dots\dots=\frac{D'O'}{DO};$$

but
$$\frac{O'A'}{P}=\frac{A'B'}{Q}=\dots\dots\dots=\frac{D'O'}{T};$$

hence
$$\frac{OA}{P}=\frac{AB}{Q}=\dots\dots\dots=\frac{DO}{T}.$$

Therefore, by Art. 36, OB represents the resultant of P and Q; OC represents the resultant of this force and R; and so on. Finally, OD represents the resultant of all the forces except T.

Now since OD is parallel to $O'D'$, it is the same line in which T acts. Therefore the resultant of all the forces except T acts in the opposite direction to T, and bears the same proportion to OD that T does.

One of the forces being equal and opposite to the resultant of all the rest, the forces are in equilibrium.

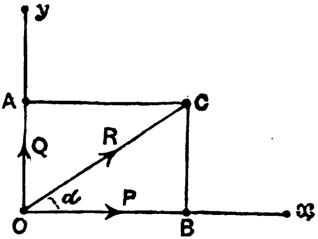
Obs.—The forces in the figure and the sides of the polygon are not necessarily all in one plane. The proposition will also hold when the sides of the polygon are inclined at any, the same, angle to the directions of the forces.

52. CONVERSE OF POLYGON OF FORCES.—The converse of the *polygon of forces* is not generally true; that is, if any number of forces meeting at a point be in equilibrium, and if a polygon be formed by drawing lines parallel to the forces, we cannot assert that the sides of the polygon shall be proportional to the forces. For, draw the polygon $OABCD$, as in Fig. Art. 51, with the sides OA, AB, BC, CD parallel and proportional to forces; OD represents the resultant of P, Q, R, S, and is therefore opposite to, and represents T in magnitude. The sides of this polygon are therefore parallel and proportional to forces. Now let $O'A'B'C'D'$ be any other polygon having its sides parallel to the forces. The two polygons are equiangular; but equiangular polygons are not necessarily similar; hence the

sides of $O'A'B'C'D'$ are not necessarily proportional to the forces.

In the particular case, when the number of sides is three, the converse is true, as we have seen in Art. 46, and because equiangular triangles are similar.

53. **Resolution of Forces.**—By the Parallelogram of Forces, any force *R*, represented by *OC*, may be resolved in any two directions, *Ox* and *Oy*, its components *P* and *Q* being determined by drawing *CA* and *CB* parallel to *Ox* and *Oy*. Thus a force may be resolved into two others, inclined at any angle, in an infinite number of ways.



In fact, the components *P* and *Q* have only to satisfy the relation

$$P^2+Q^2+2PQ\cos\theta=R^2$$

We may give the angle xOy, or θ, any value we please, and we may then determine *P* and *Q* so as to satisfy any other condition.

54. **Resolved Component of a Force.**—It is found by experiment, and may be considered also as self-evident, that *a force can produce no effect in a direction at right angles to itself.* For example, if *O* be a particle acted on by a force *P* in the direction *Ox*, and if by any means the particle be prevented from moving in any direction except along *Oy*, it is evident that the force *P* could not produce any tendency of motion in the particle along *Oy*, when the angle xOy is a right angle.

It is usual for this reason to resolve a force in two directions at right angles, as in this case neither of the components can influence the action of the other. The components are then independent of each other.

When the angle xOy is a right angle, *P* and *Q* are called the *resolved* components of *R*, and each represents the full effect of *R* in its own direction.

In this case let the angle $COB = \alpha$, then evidently

$$P = R \cos \alpha\,;$$

$$Q = R \cos (90^\circ - \alpha) = R \sin \alpha\,;$$

also $$P^2 + Q^2 = R^2.$$

Hence—*The resolved component of a force in any direction is found by multiplying the force by the cosine of the angle which it makes with the given direction.*

When CB is at right angles to Ox, OB is sometimes called the *projection* of OC on Ox, hence the resolved component of a force in any direction is the projection of the force on that direction.

55. *The resolved part of a force in any direction measures the full dynamical effect of the force in that direction.*

For, when a force R is resolved into two components P and Q at right angles, the combined effect of P and Q is the same as that of R, and Q can produce no effect in the direction of P, which is at right angles to it. Art. 54. Hence $P = R \cos \alpha$ measures the full effect of R in the direction Ox.

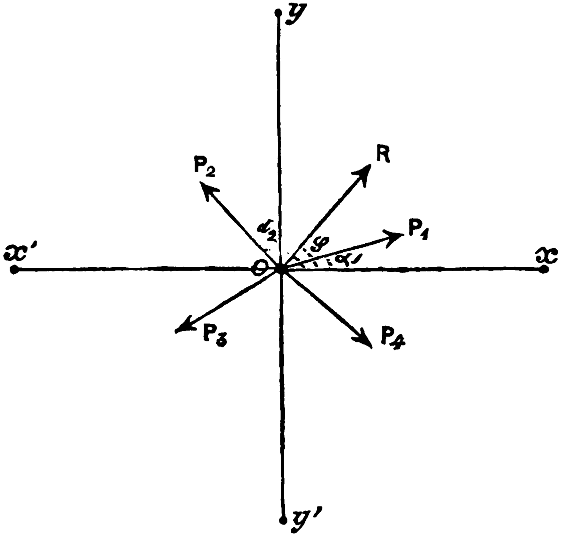
56. *The resolved part in any direction of the resultant of two forces meeting at a point is equal to the sum of the resolved parts of the components in the same direction.*

This follows from Art. 55, since the effect of a force is in every way equivalent to the combined effect of its components.

It may be also easily proved geometrically from the Parallelogram of Forces by projecting the sides and diagonal on the given direction.

This proposition may evidently be extended to any number of forces meeting at a point and lying in the same plane.

57. *To find, analytically, expressions for the magnitude and direction of the resultant of any number of forces meeting at a point, and lying in the same plane.*



Let O be the point where the forces meet; through it draw any two lines xOx' and yOy' at right angles.

Let the angles which the forces P_1, P_2 P_3, ... make with Ox measured in the direction from Ox towards Oy be a_1, a_2, a_3, ...

Now P_1 may be replaced by its components $P_1 \cos a_1$ along Ox and $P_1 \sin a_1$ along Oy. So all the forces may be replaced by their components along Ox or Ox', and along Oy or Oy'.

Hence all the forces may be replaced by two forces at right angles—one along Ox, equal to the algebraic sum of the forces in that direction, or equal to—

$$P_1 \cos a_1 + P_2 \cos a_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

the other along Oy, equal to the algebraic sum of the forces in that direction, or equal to—

$$P_1 \sin a_1 + P_2 \sin a_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

Let the former sum be denoted by $\Sigma P \cos \alpha$, and the latter by $\Sigma P \sin \alpha$; let the resultant of these two forces be R, and let its direction make an angle ϕ with Ox. Then since the forces are at right angles—

$$R^2 = (\Sigma P \cos \alpha)^2 + (\Sigma P \sin)^2 \text{............} (1)$$

$$R \cos \phi = \Sigma P \cos \alpha \text{..................} (2)$$

$$R \sin \phi = \Sigma P \sin \alpha \text{..................} (3)$$

From (2) and (3) we get by division

$$\tan \phi = \frac{\Sigma P \sin \alpha}{\Sigma P \cos \alpha} \text{............} (4)$$

These equations determine the magnitude and direction of the resultant.

Cor.—Equations (2) and (3) express the fact that the resolved part of the resultant in any direction is equal to the sum of the resolved parts of the components in the same direction.

Obs.—The Greek letter Σ is used as a convenient symbol to denote the sum of like quantities.

58. The student should remark that the *signs* of the components of any of the forces indicate the directions in which these components act from O. Thus, if P_2 lie in the second quadrant, one of its components evidently acts from O, towards y, the other from O towards x'. This is indicated by their analytical expressions. For $\sin \alpha_2$ is positive, since α_2 lies in the second quadrant; therefore $P_2 \sin \alpha_2$ is positive, which indicates that it acts from O towards y; again $\cos \alpha_2$ is negative, since α_2 lies in the second quadrant, therefore $P_2 \cos \alpha_2$ is negative, which indicates that it acts from O towards x'.

The quadrant in which the resultant acts may be determined from the equations (2) and (3), which give $\sin \phi$ and $\cos \phi$. Thus if both $\sin \phi$ and $\cos \phi$ be negative, the resultant lies in the third quadrant.

59. *To find the conditions of equilibrium of any number of forces meeting at a point, and lying in the same plain.*

When the forces are in equilibrium, we must have $R = 0$; therefore—

$$(\Sigma P \cos \alpha)^2 + (\Sigma P \sin \alpha)^2 = 0.$$

The sum of two positive quantities equal to zero, which is impossible unless each quantity is zero; therefore—

$$\Sigma P \cos \alpha = 0;$$
$$\Sigma P \sin \alpha = 0.$$

These conditions may be expressed in words thus: *The sum of the resolved parts of the forces along any direction must be zero, and the sum of the resolved parts of the forces along a perpendicular direction must be also zero.*

60. *If the sums of the resolved parts of the forces along any two directions whatsoever be severally zero, the forces are in equilibrium.*

Let one direction be Ox, the other a line inclined to Ox at an angle θ.

The angles which the forces make with Ox are α_1, α_2 ....therefore the angles which they make with the second direction are $(\alpha_1 - \theta), (\alpha_2 - \theta)$..............hence by hypothesis

$$P_1 \cos (\alpha_1 - \theta) + P_2 \cos (\alpha_2 - \theta) + \dots\dots\dots\dots = 0,$$

or $$P_1 (\cos \alpha_1 \cos \theta + \sin \alpha_1 \sin \theta) + \dots\dots\dots\dots = 0;$$

$$\therefore \cos \theta \,(P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \dots\dots\dots\dots)$$
$$+ \sin \theta \,(P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \dots\dots\dots\dots) = 0;$$

but by hypothesis—

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \dots\dots\dots\dots = 0;$$

therefore $$P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \dots\dots\dots\dots = 0.$$

Hence the conditions of Art. 59 are satisfied, and therefore the forces are in equilibrium.

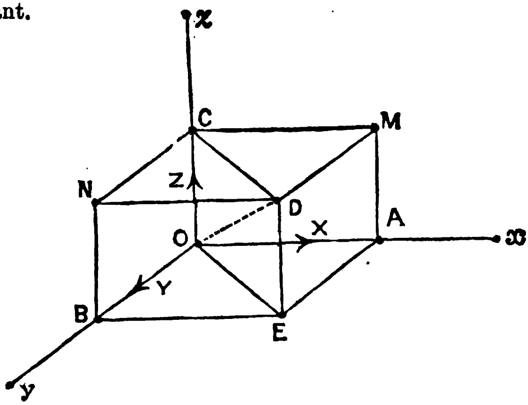
The following article is an easy extension of the Parallelogram of Forces.

61. PARALLELOPIPED OF FORCES.—*If three forces, meeting at a point, but not lying in the same plane, be represented in direction and magnitude by three lines drawn from the point, their resultant will be represented in direction and magnitude by*

the diagonal, of the parallelopiped, drawn from the point, of which the three lines are adjacent edges.

Let OA, OB, OC represent the forces acting at O.

Complete the parallelopiped; then OD represents their resultant.



For, the resultant of two of the forces X and Y, represented by OA and OB, is represented by OE; they may be replaced by it. Again the resultant of this force and Z is represented by OD, which therefore represents the resultant of the three forces X, Y, Z.

Cor. 1.—If the parallelopiped be rectangular, we have

$$OE^2 = OB^2+OA^2,\ OD^2 = OE^2+OC^2;$$

$$\therefore OD^2 = OA^2+OB^2+OC^2,$$

or

$$R^2 = X^2+Y^2+Z^2.$$

Cor. 2.—Again if the angles DOA, DOB, DOC be denoted by α, β, γ, respectively, $OA = OD \cos \alpha$, $OB = OD \cos \beta$, $OC = OD \cos \gamma$. $X = R \cos \alpha$, $Y = R \cos \beta$, $Z = R \cos \gamma$: thus X, Y, Z are the resolved components of R along the three directions.

Cor. 3.—Squaring and adding these equations

$$OD^2 (\cos^2 \alpha+\cos^2 \beta+\cos^2 \gamma) = OA^2+OB^2+OC^2$$

$$\therefore \cos^2 \alpha+\cos^2\beta+\cos^2 \gamma = 1.$$

62. *To find the direction and magnitude of the resultant of any number of forces meeting at a point, but not lying in the same plane.*

Draw through the point any three lines mutually at right angles; let the angles which the directions of the forces make with these lines be α_1, β_1, γ_1; α_2, β_2, γ_2; ....

Let the resultant make angles with these lines a, b, c.

Then, as in Art. 57, the forces may be replaced by their components along Ox, Oy, and Oz, and are hence equivalent to three forces whose directions are mutually at right angles. Thus—

$$R^2 = (\Sigma P \cos \alpha)^2 + (\Sigma P \cos \beta)^2 + (\Sigma P \cos \gamma)^2;$$

$$R \cos a = \Sigma P \cos \alpha;$$

$$R \cos b = \Sigma P \cos \beta;$$

$$R \cos c = \Sigma P \cos \gamma.$$

These equations are sufficient to determine the resultant and the angles which it makes with Ox, Oy, Oz.

Cor.—The conditions of equilibrium are found as in Art. 59 by making $R = 0$; therefore

$$\Sigma P \cos \alpha = 0, \ \Sigma P \cos \beta = 0, \ \Sigma P \cos \gamma = 0$$

These conditions may be expressed in words thus: *The sums of the resolved parts of the forces along any three directions, mutually at right angles, must be severally equal to zero.*

63. *If three forces, lying in the same plane, and not parallel, keep a body at rest, their directions must meet at a point.*

For, the directions of any two of the forces meet at a point: they may be replaced by their resultant acting at that point, and the third force cannot counteract its effect, except it be equal and opposite.

64. The conditions of equilibrium of three forces which are not parallel, but which lie in one plane, may be expressed as follows:—

(1.) *The directions of the three forces must meet at a point.*

(2.) *Any one of the forces must be equal and opposite to the resultant of the other two.*

It will be shown hereafter that if three forces keep a rigid body in equilibrium, their directions must lie in the same plane.

We will now give an example of the equilibrium of three forces acting on a rigid body.

65. EXAMPLE.—*Three forces acting in one plane on a rigid body have directions perpendicular to the sides of a given triangle, and each force is proportional to the side perpendicular to which it acts; show that if the forces all act inwards, or all act outwards, they will keep the body in equilibrium.*

Let ABC be the given triangle; P, Q, R the forces acting perpendicularly to the sides BC, CA, AB through their middle points.

Lines perpendicular to the sides of a triangle through their middle points meet at the centre of the circle circumscribing the triangle. Euclid IV, 5.

Hence the first condition of equilibrium of Art. 64 is satisfied.

Again, suppose the triangle ABC to be turned round in its own plane through a right angle. The forces would then be parallel and proportional to the sides of a triangle taken in order. Hence, by the *Triangle of Forces*, they equilibrate.

The second condition of Art. 64 may otherwise be shown to be satisfied by applying Art. 42.

The angles between the forces are evidently the supplements of the angles of the triangle.

Therefore, $\sin A = \sin (QR)$ &c., &c.

By hypothesis
$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c};$$

but
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

therefore
$$\frac{P}{\sin (QR)} = \frac{Q}{\sin (PR)} = \frac{R}{\sin (PQ)}$$

Hence the forces equilibrate.

The conditions of equilibrium of Art. 59 might also be applied.

Resolve the forces parallel to any side BC of the triangle. The sum of resolved parts is $Q \sin C - R \sin B$.

Since $Q : R = b : c$, this sum vanishes. Similarly, the sum of resolved parts parallel to any other side vanishes. Hence, by Art. 60, the forces equilibrate.

EXAMPLES.

1. Two forces of 30 and 40lbs., respectively, meeting at a point, are at right angles to each other; find their resultant.

Ans. 50lbs.

2. The resultant of two forces, acting at a right angle, is 56 units, and one of the component is 31 units; find the other.

Ans. 46.6.

3. Two forces, one of which is double the other, meet at a point, and are such that if 4 units of force be added to the larger, and the smaller be doubled, the direction of the resultant is unchanged; find the forces.

Ans. 2 and 4.

4. Two forces act on a particle, and their greatest and least resultants are 88lbs. and 24lbs., respectively; find the forces.

Ans. 56lbs. and 32lbs.

5. Two equal forces act at the same point, making with each other an angle of 120°; find the magnitude and direction of their resultant.

Ans. The resultant bisects the angle, and is equal to either component.

6. The resultant of two forces acting at right angles is double the lesser of the components; find the direction of the resultant.

Ans. It makes an angle of 60° with the lesser component.

7. Two forces, each equal to P, act at an angle of 60°; find their resultant.

Ans. $P\sqrt{3}$.

8. At what angle should two forces of 3lbs. and 4lbs act, in order that their resultant should be $\sqrt{37}$ lbs.?

Ans. 60°.

9. One of two forces meeting at a point is double the other, and their resultant is equal to half their sum; find the angle between them.

Ans. $\overset{-1}{\cos}\left(-\frac{11}{16}\right)$.

10. Can a particle be kept at rest by three forces, whose magnitudes are as the numbers 2, 3 and 6?

Ans. No.

11. Can a particle be kept at rest by three forces, whose magnitudes are as the numbers 2, 3 and 5?

Ans. Yes.

12. Forces proportional to the numbers 1, 2 and $\sqrt{3}$ keep a particle at rest; determine the angles between their direction.

Ans. 90°, 120°, 150°.

13. Three forces meeting at a point equilibrate, and are proportional to $\sqrt{3}+1$, $\sqrt{6}$ and 2; find the angles between their directions.

Ans. 105°, 120°, 135°.

14. Two forces of 72lbs. and 108lbs. meet at a point, and the angle between them is 60°; find the angles into which the resultant divides this angle, having given—

$\log 2 = .30103$; $\log 3 = .47712$;
$L \tan 6^\circ 35' = 9.06224$; diff. for $1' = .00111$.

Ans. 36° 35′ 12″; 23° 24′ 48″.

15. The resultant of two forces meeting at a point makes angles of 15° and 45° with the components, and the larger component is 36lbs.; find the resultant, having given log 2, log 3, and
$\log 4.409 = .64434$.

Ans. 44.09lbs.

16. Forces of 12, 18 and 24 units, respectively, equilibrate at a point; find the angle between the forces 12 and 18, having given log 2, log 3 and
$L \tan 52^\circ 14' = 10.11084$; diff. for $1' = .00026$.

Ans. 75° 31′ 22″.

17. Three forces of 99, 100 and 101 units, respectively, act at a point and are inclined to each other at angles of 120°; find the magnitude of their resultant, and its inclination to the force of 100 units.

Ans. $\sqrt{3}$; 90°.

18. Four forces of 1, 2, 3, 4 units act from the centre of a square towards the angular points; find magnitude and direction of resultant.

Ans. $2\sqrt{2}$. It bisects the angle between the forces 3 and 4.

19. Six forces of 1, 2, 3, 4, 5, 6 units act from the centre of a regular hexagon towards the angular points; find the magnitude and direction of resultant.

Ans. 6 in the direction of the force 5.

20. Four forces, each equal to P, act at a point; the angle between the first and second, that between the second and third, and that between the third and fourth are each equal to 30°; find the magnitude of resultant.

Result. $P\sqrt{6+3\sqrt{3}}$.

21. Three forces of 1, 2 and 3 units, respectively, act at a point; the angle between the first and second is 60°, the angle between the second and third is 30°; find the angle which the resultant makes with the first of the given forces.

Result. $\tan^{-1}\frac{3+\sqrt{3}}{2}$.

22. Forces are represented by two chords of a circle, at right angles to each other, drawn from a fixed point on the circumference; prove that their result is constant in magnitude and direction.

23. A and B are fixed points, and O a variable point; at O forces act, represented by OA and OB; prove that if their resultant be of constant magnitude, the locus of O will be a circle.

24. Four forces are represented in magnitude and direction by lines drawn from a point to the angular points of a parallelogram; prove that their resultant is represented, in direction, by the line joining the point to the intersection of diagonals, and, in magnitude, by four times that line.

25. Three forces are represented in magnitude and direction by lines drawn from a point to the vertices of a triangle; prove that their resultant is represented, in direction, by the line joining the point to the intersection of bisectors of sides (the centre of gravity of triangle) and, in magnitude, by three times that line.

26. *ABCD* is a quadrilateral; forces are represented, in magnitude and direction, by *AB*, *AD*, *CB*, *CD*; show that their resultant is represented, in direction, by the line joining the middle points of the diagonals, and, in magnitude, by four times that line.

27. Four forces are represented by lines drawn from a point to the angular points of a quadrilateral; prove that their resultant is represented by four times the line joining the point to the intersection of lines joining the middle points of opposite sides.

28. Three forces are represented by lines drawn from a point to the vertices of a triangle; show that they are equivalent to three forces represented by the lines joining the point to the middle points of the sides of triangle.

29. Three forces are represented by three chords of a sphere, mutually at right angles, drawn from a point on the sphere; prove that their resultant is represented by the diameter of the sphere drawn through the point.

30. A body whose weight is 5lbs. is suspended freely from a fixed point by a flexible string; find what horizontal force applied to the string will draw the upper portion of it 30° out of the vertical.

Ans. 2.886lbs.

31. A boat is dragged along a stream 30 yards wide, by a man on one bank, the length of the rope from the boat to the man being 50 yards. The boat is steered straight down the middle of the stream; determine the effective force in that direction, if the rope be pulled with a force of 70lbs.

Ans. 56lbs.

32. A body whose weight is 6lbs is suspended by two strings of equal length, the other ends of the strings being attached to two fixed points in a horizontal line, at a distance a part equal to the length of either string; find the tension of either string.

Ans. $2\sqrt{3}$ lbs.

33. A body whose weight is 25lbs. hangs at rest, attached to the ends of two strings, the lengths of which are 3 and 4 feet, and the other ends of the strings are fastened at two points in a horizontal line distant 5 feet from each other; find the tension of each string.

Ans. 15lbs.; 20lbs.

34. A cord, whose length is $2l$, is fastened at two points in the same horizontal line, at a distance $2a$ apart. A smooth ring upon the cord sustains a body, whose weight is W; find the tension of the cord.

Result. $\dfrac{Wl}{2\sqrt{l^2-a^2}}$.

35. Three forces are represented, in magnitude and direction, by lines drawn from the vertices of a triangle to the points of bisection of the opposite sides; prove that they equilibrate.

36. Three forces act from the angular points of a triangle in directions perpendicular to the opposite sides, and each force is inversely proportional to the perpendicular along which it acts; show that the forces equilibrate.

37. Three forces act from the centre of the circle circumscribing a given triangle towards the angular points; find the conditions of equilibrium.

Result. $\dfrac{P}{\sin 2A} = \dfrac{Q}{\sin 2B} = \dfrac{R}{\sin 2C}$.

38. Forces act along the bisectors of the angles of a given triangle, and are proportional to $\cos \frac{1}{2}A$, $\cos \frac{1}{2}B$, $\cos \frac{1}{2}C$; prove that they equilibrate.

39. Forces act perpendicularly at the middle points of the sides of a polygon, inscribable in a circle, and each force is proportional to the side perpendicular to which it acts; prove that the forces equilibrate.

40. Assuming the truth of the Parallelogram of Forces for the *magnitude*, prove it also for the *direction* of the resultant.

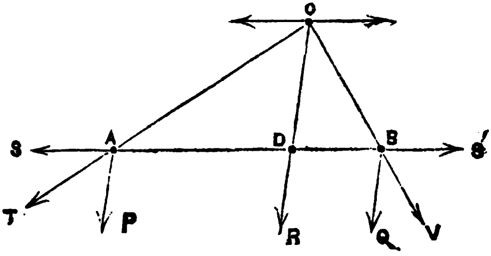
CHAPTER III.

PARALLEL FORCES.

66. Forces whose lines of action are parallel are called *parallel forces ;* those which tend parallel to the same direction may conveniently be called *like,* and those which tend parallel to the opposite direction, *unlike.*

67. *The resultant of two like parallel forces acting on a rigid body is a like parallel force equal to their sum, and whose direction divides any line drawn across the forces, internally, in their inverse ratio.*

Let AB be the line joining any two points A and B on the lines of action of the forces P and Q. The forces may be supposed to act at these points. Art. 19.



At A and B apply two equal forces, S and S', acting along AB in opposite directions. This will not affect the action of the forces P and Q. Art. 18.

Replace P and S at A by their resultant T, and Q and S' by their resultant V; let the lines of action of T and V meet at C; T and V may be transferred so as to act at C, C being supposed to be rigidly connected with A and B. Draw CD parallel to the direction of P and Q meeting AB in D.

Let the forces T and V at C be resolved into their original components parallel to their original directions. Thus the effect of the forces P and Q at A and B is equivalent to that of two equal and opposite forces S and S' at C, which equilibrate, and may hence be removed, and two forces, P and Q acting from C along CD.

Hence the resultant of the like parallel forces P and Q is a parallel force $P+Q$, acting from C along CD, and it may be supposed to act at D

Again, to find the point D, since the sides of the triangle ADC are parallel to the forces P, S and their resultant T, we have, by Art. 44,

$$\frac{CD}{DA}=\frac{P}{S};$$

similarly $$\frac{DB}{DC}=\frac{S'}{Q};$$

therefore, since $S=S'$, $$\frac{BD}{DA}=\frac{P}{Q}.$$

Thus the line of action of the resultant divides AB, internally, into segments which are inversely proportional to the forces at A and B, respectively.

68. Given the length of $AB=a$, to calculate the segments x and y made by resultant.

We have $$\frac{AD}{BD}=\frac{Q}{P};$$

therefore $$\frac{AD}{AD+BD}=\frac{Q}{P+Q},$$

$$AD=x=\frac{Qa}{P+Q},$$

similarly $$BD=y=\frac{Pa}{P+Q},$$

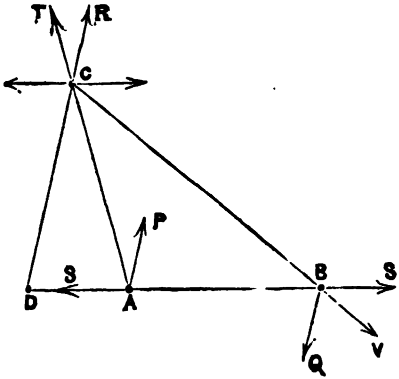
since $$P+Q=R$$

we have $$R.\,x=Q.\,a;$$

$$R.\,y=P.\,a;$$

also $$P.\,x=Q.\,y.$$

69. *The resultant of two unlike parallel forces is a parallel force equal to their difference, acting in the direction of the greater, and whose line of action divides any line drawn across them, externally, in their inverse ratio.*



Let P be greater than Q. The student will have no difficulty in proving, as in Art. 67, that the resultant of the unlike forces P and Q is a parallel force, in the same plane, $P-Q$ acting in the direction of P, in a straight line which cuts AB externally at D, so that—

$$\frac{AD}{BD}=\frac{Q}{P}$$

70. Given the length $AB=a$, to calculate the segments x and y made by resultant.

We have $$\frac{AD}{BD}=\frac{Q}{P},$$

therefore $$\frac{AD}{BD-AD}=\frac{Q}{P-Q},$$

$$AD=x=\frac{Qa}{P-Q},$$

similarly $$BD=y=\frac{Pa}{P-Q},$$

since $$P-Q=R,$$
we have $$R.\,x=Q.\,a;$$
$$R.\,y=P.\,a;$$
also $$P.\,x=Q.\,y.$$

These formulæ may be got from those of Art. 68 by changing Q into $-Q$.

71. CENTRE OF PARALLEL FORCES.—If any two points, A and B, on the lines of P and Q, be taken as their points of application, then, no matter how the forces P and Q be turned round these points, but so as always to remain parallel to each other, their resultant will always pass through the point D, which divides AB, internally or externally, according as the forces are like or unlike, in the inverse ratio of the forces.

For this reason the point D is sometimes called the *centre* of the parallel forces P and Q acting at A and B.

72. *Conditions of equilibrium when three parallel forces act on a rigid body.*

Any one of the forces must have the same line of action as the resultant of the other two; it must be equal to the resultant in magnitude, and opposite in way of action.

Hence the forces must lie in the same plane; one of them must be unlike the other two, and equal to their sum; and its line of action must lie between theirs, dividing the distance between them in the inverse ratio of the forces.

73. Assuming the rules for the composition of two like parallel forces, Art. 67, it is easy to deduce the rules of composition when the forces are unlike. Art. 69.

Let the like parallel forces P and Q act at A and B, and let D be the point of application of their resultant $P+Q$.

Apply at D a force R equal and opposite to the resultant of P and Q, then the forces P, Q, R, are in equilibrium; hence P is equal and opposite to the resultant of R and Q. Hence the resultant of R and Q is a parallel force $R-Q=P$ acting at A in the direction of R.

Again $$\frac{DB}{AD}=\frac{P}{Q};$$

therefore $$\frac{DB+AD}{AD}=\frac{P+Q}{Q};$$

hence $$\frac{AB}{AD}=\frac{R}{Q}.$$

Hence the resultant of R and Q divides BD externally into parts which are inversely as the forces at D and B.

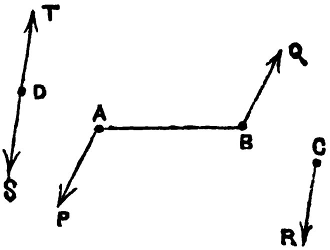
74. When two unlike parallel forces are equal in magnitude, it is impossible to find a single resultant. The process of Art. 69 fails in this case. In fact, the addition of the equal forces S and S' to the equal and unlike forces P and Q gives still two equal and unlike parallel forces T and V.

It will be also seen that the formulæ of Art. 70 lead to a nugatory result. Since $P = Q$ gives $R = 0$ and $x = \infty$.

Def.—A system of two equal unlike parallel forces is called a *couple*. It will be shown in the next article, by a different method, that such a pair of forces cannot have a *single* resultant.

75. *Two equal and unlike parallel forces cannot have a single resultant.*

If possible, let the equal and unlike parallel forces P and Q have a resultant acting at any point C. At C apply a force R equal and opposite to the resultant of P and Q. The three forces, P, Q and R, are in equilibrium.



Find the point D, situated with regard to the points A, and B in the same manner that C is with regard to B and A. Apply at D in opposite directions two forces T and S, each equal and parallel to R. This will not disturb the equilibrium. Hence the five forces, P, Q, R, S, T, are in equilibrium.

But since P, Q, R are in equilibrium, by *symmetry*, Q, P, T are in equilibrium, and may be removed without disturbing the equilibrium.

There remain R and S in equilibrium. But this is impossible, since they are like and parallel. Hence the forces P and Q cannot have a single resultant.

Obs.—The physical effect of a couple will be considered further on.

76. By the converse of Arts. 67 and 69, a force R may be resolved into two parallel forces, P and Q, in an infinite number of ways.

The positions and magnitudes of the components have only to satisfy the following conditions.

The lines of action of the two components, P and Q, must be in the same plane with that of R; their *algebraic* sum must be equal to R; and the position of R must divide the distance between them in their inverse ratio.

77. *To determine the magnitude and position of the resultant of any number of parallel forces acting on a rigid body.*

Let the forces be P_1, P_2, P_3 .... P_n, and let A, B, C .... be any points on their lines of action.

Join AB and divide it at M, either internally or externally, according as the forces P_1 and P_2 are in same or opposite directions, so that

$$\frac{AM}{BM}=\frac{P_2}{P_1}$$

The resultant of P_1 and P_2 is P_1+P_2 acting parallel through M.

Join MC and divide it at N, so that—

$$\frac{MN}{CN}=\frac{P_3}{P_1+P_3}$$

The resultant of P_1+P_2 and P_3, that is, of P_1, P_2, P_3 is $P_1+P_2+P_3$ acting parallel through N.

Proceeding in this manner we at last find the point O through which the resultant of all the forces, ΣP, acts parallel to the components.

The algebraic sign of ΣP determines the direction, parallel to the components, to which the resultant tends, forces tending in one direction being considered positive, and forces parallel to the opposite direction, negative.

78. CENTRE OF PARALLEL FORCES.—The point O thus found possesses the property that no matter how the forces $P_1, P_2, P_3,$ .... be turned round the points $A, B, C,$ .... , but so as always to remain parallel amongst themselves, their resultant always passes through O. This

point is called the *centre* of the parallel forces P_1, P_2, P_3 .... acting at A, B, C ....

It is evident that the position of O depends on what points A, B, C, ... on the lines of action of the forces are chosen as their points of application, but, if different points be taken, O must always lie on the line of action of the resultant, which is thus the *locus* of O.

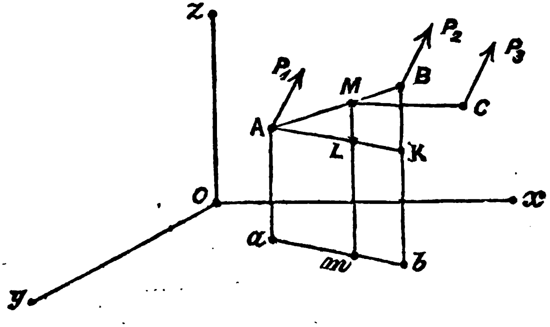
If in the process explained in Art. 77 we arrived finally at two equal parallel forces acting in opposite directions not in the same straight line, the system of forces would be equivalent to a couple and could have no single resultant, or centre. Art. 75.

It is evident that a system of *like* parallel forces must always have a single resultant and a centre.

In the next article it will be shown how the centre of parallel forces can be determined analytically.

79. *To determine analytically the position and magnitude of the resultant of any number of parallel forces acting on a rigid body.*

Let the forces be P_1, P_2, P_3 ...., and let A, B, C, ... be points on their lines of action. Take any plane xOy, and drop perpendiculars from A, B, C, ... on the plane xOy; let the lengths of these perpendiculars be z_1, z_2, z_3 .... In the figure two only of these perpendiculars are drawn, $A\,a = z_1$ and $B\,b = z_2$.



Divide AB at M inversely as the forces P_1 and P_2; draw ALK parallel to amb, and Mm perpendicular to the plane xOy; let $Mm = \alpha$, then—

$$\frac{AM}{BM} = \frac{P_2}{P_1};$$

$$\frac{AM}{AM+BM} = \frac{AM}{AB} = \frac{P_2}{P_1+P_2}.$$

But by similar triangles we have

$$\frac{AM}{AB} = \frac{ML}{BK} = \frac{\alpha - z_1}{z_2 - z_1};$$

therefore
$$\frac{\alpha - z_1}{z_2 - z_1} = \frac{P_2}{P_1+P_2},$$

from which
$$\alpha = \frac{P_1 z_1 + P_2 z_2}{P_1 + P_2}.$$

This formula determines the distance of the point of application of the resultant of two forces from the plane xOy.

Again P_1 and P_2 may be replaced by a single force equal to P_1+P_2 acting parallel through M; we can find by the above formula the distance of the point of application of the resultant of this force and P_3 from the plane xOy; let the distance be β; then—

$$\beta = \frac{(P_1+P_2)\,\alpha + P_3 z_3}{(P_1+P_2)+P_3}$$

substituting for α its value we get

$$\beta = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3}{P_1+P_2+P_3}$$

and the three forces are equivalent to a single force $P_1+P_2+P_3$.

This process may in a similar manner be extended to any number of parallel forces. Let $\bar{z}$ be the distance of the point of application of the final resultant from the plane.

$$\bar{z} = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \cdots\cdots\cdots}{P_1+P_2+P_3+\cdots\cdots\cdots}.$$

Let yOz and zOx be two other planes so taken that the three planes shall be mutually at right angles, and let distances from these planes be denoted by x and y symbols respectively, thus—

$$\bar{x}=\frac{\Sigma Px}{\Sigma P}\ ;\ \bar{y}=\frac{\Sigma Py}{\Sigma P}\ ;\ \bar{z}=\frac{\Sigma Pz}{\Sigma P}.$$

$$R=\Sigma P.$$

These equations determine the magnitude of the resultant, and a point on its line of action; hence its position.

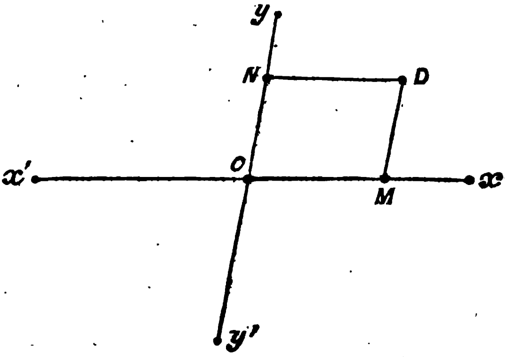
Cor. 1.—The values of $\bar{x}$, $\bar{y}$, $\bar{z}$ depend only on the algebraic magnitudes of the forces and their points of application. Hence, if the forces be all turned round their points of application so as always to remain parallel, the resultant must always pass through the point so determined, which point is the centre of parallel forces.

Cor. 2.—From the formulæ which determine the centre, it is evident that a system of *like* parallel forces acting at definite points has got one and only one centre.

80. For the convenience of the student who is unacquainted with the elements of analytical geometry, it seems desirable to make a few remarks regarding the methods of determining the position of a point in a plane, and in space.

To determine the position of a point in a plane

Take in the plane any two lines xOx' and yOy' intersecting in O. Let D be any point in the plane. Draw DM and DN parallel to Oy and Ox. Let $OM=DN=x$; and $ON=DM=y$.



Then if x and y be given the position of D is determined. For, to find D we have only to take $OM = x$, $ON = y$, and to draw through M and N parallels to Oy and Ox, respectively, their intersection is the point required.

x and y are called the *co-ordinates* of the point D, xOx', yOy' are called the axes of co-ordinates, and O is called the origin of co-ordinates.

As in Trigonometry, x is considered positive or negative according as it is measured from O along Ox, or along Ox'; y is considered positive or negative according as it is measured from O along Oy or along Oy'.

The axes are generally taken at right angles; the co-ordinates are then called rectangular co-ordinates.

To determine the position of a point in space.

Take any three planes xOy, yOz, and zOx intersecting in O, Fig. Art. 61. Let D be any point. From D draw DN, DM, DE, parallel to Ox, Oy, Oz, respectively, to meet the planes. Let $OA = DN = x$; $OB = DM = y$; $OC = DE = z$.

Then if x, y, z, be given, the position of D is determined. For, to find D, we have only to take $OA = x$, $OB = y$, $OC = z$, and to draw through A, B, C, planes parallel to the planes yOz, zOx, xOy, respectively; their intersection is the point required.

x, y, and z are called the co-ordinates of the point D, and xOy, yOz and zOx are the co-ordinate planes; Ox, Oy, Oz are called the axes of co-ordinates, and O the origin. Similar remarks apply to the signs of the co-ordinates x, y, z.

The co-ordinate planes are generally taken at right angles to each other, in which case the axes of co-ordinates are also mutually at right angles.

81. REMARKS ON THE FORMULÆ OF ART. 79.—In the formulæ of Art. 79 z_1, z_2, ... $\bar{z}$ are perpendicular distances on the plane xOy. The proof, however, remains the same when these distances are all measured parallel to a given direction such as Oz. The formulæ are thus true for oblique as well as rectangular co-ordinates. We now give some special cases of these formulæ. It will be observed that any point on the line of action of the resultant is sufficient to determine its position.

I. Suppose the points where the lines of action of the forces meet a given plane, are known. We have only to take two axes in this plane, Ox and Oy; then $z_1 = z_2 = \dots\dots\dots = \bar{z} = 0$, and the point where the line of the resultant cuts this plane is given by

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}; \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}.$$

II. When the forces all lie in the same plane, we may take any two axes in this plane, and if the co-ordinates of the points of application of the forces be known, the co-ordinates of the centre of parallel forces are given by

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}; \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}.$$

III. When the forces all lie in one plane, if we know the points where their lines of action cut any line Ox, the point where the line of the resultant cuts this line is determined by

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}.$$

The student should draw diagrams and *write* out the proof of the formulæ for each of these special cases.

EXAMPLES.

1. Two like parallel forces of 5 and 13lbs. respectively have their points of application 6 feet apart; find the point of application and magnitude of resultant.

Ans. Magnitude 18lbs. Point of application $4\frac{1}{3}$ feet from the force of 5lbs.

2. Let the forces be 23lbs. and 42lbs., and their points of application 14 feet apart.

Ans. 65lbs. at a distance 9.047 feet from the force of 42lbs.

3. Let the forces be 126 and 220, and their points of application 26 inches apart.

Ans. 346 at a distance of 9.47 inches from force 126.

4. A body is acted on by two unlike parallel forces, $2P$ and $7P$, 10 inches apart; determine the magnitude, position, and direction of their resultant.

Ans. Magnitude, $5P$; direction, the same as that of the force $7P$; point of application, 4 inches from $7P$ and 14 inches from $2P$.

5. Four like parallel forces act at the angles of a plane quadrilateral, and are inversely proportional to the segments of its diagonals nearest to them; show that their centre is at the intersection of diagonals.

6. ABC is a triangle, and O any point within it; like parallel forces act at A and B, proportional to the areas BOC and ACO respectively; show that their resultant acts at the point where CO produced meets AB.

7. ABC is a triangle, and O any point within it; like parallel forces act at A, B, and C, and are proportional to the areas BOC, COA, and AOB respectively: show that their resultant acts at O.

8. If the point O in the last example be outside the triangle, between AB and AC produced, and if the force at A be unlike the forces at B and C, show that the resultant acts at O.

9. ABC is a triangle, and parallel forces act at B and C, each proportional to the opposite side of triangle; prove that their resultant acts at the point where the internal or external bisector of the vertical angle A cuts the base, according as the forces are like or unlike.

10. Like parallel forces act at the angles A, B, C of a triangle, and are respectively proportional to a, b, c; show that their resultant acts at the centre of inscribed circle.

11. If in the last example the force acting at A be unlike those at B and C, show that the resultant acts at the centre of circle escribed to a.

12. Equal and like parallel forces act at the angles of a triangle; show that their centre is the point of intersection of lines drawn from the vertices bisecting the opposite sides.

13. Like parallel forces act at five of the angular points of a regular hexagon; show that the centre of parallel forces is on the diameter of circumscribing circle passing through the sixth angular point and at a distance from it, $\frac{6}{5}r$.

14. At each of $n-1$ of the angular points of a regular polygon of n sides equal and like parallel forces act; find the distance of their centre from the centre of circumscribing circle.

Solution. Let A be the vertex at which no force acts; O the centre of circle, and C the centre of parallel forces. C evidently lies on AO produced. If an equal and like parallel force, P, acted at A, the resultant of the n forces would evidently act through the centre. Hence nP at O would be the resultant of P at A and $(n-1)P$ at C, therefore

$$\frac{OC}{OA}=\frac{P}{(n-1)P}; \ OC=\frac{r}{n-1}.$$

15. Find the centre of equal and like parallel forces acting at seven of the angular points of a cube whose side is a.

Ans. The centre lies on the diagonal, through the eighth angular point, at a distance from centre of cube $a\sqrt{3} \div 14$.

16. Parallel and like forces of 4, 2, 6, and 10 units respectively lie in the same plane, and are one foot apart from each other; find position and magnitude of resultant.

Ans. Magnitude 22. Position the same as that of force 6.

17. Parallel and like forces, P, $2P$, $3P$, and $4P$, act on a straight rod at distances from one end of 1, 2, 3, and 4 feet respectively; find the distance of resultant from that end.

Ans. 3 feet.

18. Parallel and like forces, proportional to $1, 2, 3 \ldots\ldots\ldots n$, act on a rod at distances from one end of $1, 2, 3, \ldots\ldots\ldots n$ feet respectively; find the distance of resultant from that end.

Ans. $\frac{2n+1}{3}$ ft.

19. Parallel forces $+1, -2, +3, -4$ act in the same plane on a rigid body. Their lines of action cut a line Ox drawn in their plane at distances from O of 1, 2, 3, 4 feet respectively; determine the position and magnitude of the force which would keep the body at rest.

Ans. A parallel force $+2$, as a distance 5 feet from O, along Ox.

20. Equal and like parallel forces act at the angles of a triangle ABC, given the co-ordinates of A, B, and C, with respect to any axes in the plane of triangle; find the co-ordinates of centre of parallel forces.

Ans. $\bar{x} = \frac{1}{3}(x_1 + x_2 + x_3)$;
$\bar{y} = \frac{1}{3}(y_1 + y_2 + y_3)$.

21. Equal and like parallel forces act at the angular points A, B, C, D, of a triangular pyramid: given the co-ordinates of A, B, C, D, with respect to any three planes; find the co-ordinates of the centre of parallel forces.

Ans. $\bar{x} = \frac{1}{4}(x_1 + x_2 + x_3 + x_4)$;
$\bar{y} = \frac{1}{4}(y_1 + y_2 + y_3 + y_4)$;
$\bar{z} = \frac{1}{4}(z_1 + z_2 + z_3 + z_4)$.

22. Parallel forces P, Q, R act at the angular points A, B, C of a triangle; the perpendicular distances of their centre from the sides a, b, c are α, β, γ respectively, and the area of triangle is Δ; prove that

$$\frac{a\alpha}{P} = \frac{b\beta}{Q} = \frac{c\gamma}{R} = \frac{2\Delta}{P+Q+R}.$$

23. When the centre of parallel forces conicides with the centre of inscribed circle, find the relation between forces.

Ans. $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$.

24. In the same case, when the centre of parallel forces coincides with the centre of circumscribing circle; find the relation between forces.

Result. $\frac{P}{a \cos A} = \frac{Q}{b \cos B} = \frac{R}{c \cos C}$.

25. In the same case, when the centre of parallel forces coincides with the intersectien of perpendiculars from vertices on the opposite sides (the orthocentre of triangle); find the relation between the forces.

Result. $\frac{P}{\tan A} = \frac{Q}{\tan B} = \frac{R}{\tan C}$.

26. Parallel forces P, Q, R, S act at the angular points A, B, C, D of a triangular pyramid: the perpendicular distances of

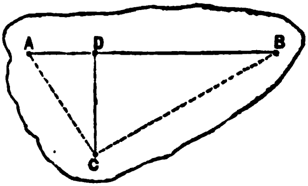
their centre from the planes BCD, ACD, ABD, ABC, are α, β, γ, δ respectively; the areas of triangles BCD, ACD, ABD, ABC, are Δ_1, Δ_2, Δ_3, Δ_4, and the volume of pyramid is V; prove that

$$\frac{\alpha\,\Delta_1}{P}=\frac{\beta\,\Delta_2}{Q}=\frac{\gamma\,\Delta_3}{R}=\frac{\delta\,\Delta_4}{S}=\frac{3V}{P+Q+R+S}.$$

CHAPTER IV.

MOMENTS AND CONDITIONS OF EQUILIBRIUM.

82. *Def.*—The *moment* of a force with respect to a point is the product of the force and the perpendicular drawn on its line of action from the point.



Let AB be the line of action of a force P; $CD = p$, the perpendicular from any point C on AB; the moment of P with respect to C is Pp. In this product the abstract number which represents the magnitude of the force P is multiplied by the number which represents the length of the perpendicular.

83. *The magnitude of the moment of a force with respect to any point is represented geometrically by twice the area of triangle whose base is the line which represents the force, in magnitude and position, and whose vertex is the point.*

For, let AB represent the magnitude, as well as position of the force P; that is, AB contains as many units of length as P contains units of force; then 2area $CAB = AB \cdot CD = Pp =$ moment of P with respect to C.

It is evident that the moment of a force with respect to a point never vanishes except when the point lies on the line of action of the force.

84. Moments Positive or Negative.—Suppose a force P to act on a body rigidly connected with a fixed point C. It is evident that the effect of the force on the body would be to produce, or tend to produce, rotation round an axis through C perpendicular to the plane CAB. Fig. Art. 82. We shall afterwards see that the moment of P with respect to C is the mathematical measure of this tendency to produce rotation.

If the force P acted from A towards B the tendency would be to produce rotation in the direction of the hands of a watch. But if the force P acted from B towards A the tendency of rotation would be in an opposite direction.

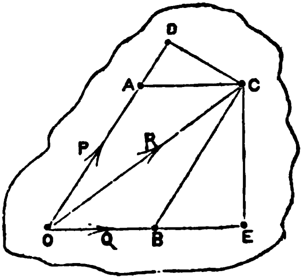
Moments of forces with respect to any point are considered positive or negative according as the forces tend to turn a body in the same or in opposite directions round the point. It is indifferent which direction of rotation is chosen for the positive one.

85. *The algebraic sum of the moments of any two forces, which have got a single resultant, with respect to any point on the line of action of the resultant, is equal to zero.*

First, let the forces P and Q meet at a point O, and let C be any point on the line of action of their resultant.

From C draw CA and CB parallel to the lines of the forces Q and P respectively.

Then by the *parallelogram of forces* OA and OB represent the forces P and Q. Let the perpendiculars $CD = p$; $CE = q$.



Then $Pp = 2$ area OAC; $Qq = 2$ area OBC;

but area $OAC =$ area OBC,

and if the moment Pp be considered positive, Qq is negative, since the forces evidently tend to produce rotation in opposite directions round C,

therefore $Pp + Qq = 0.$

Next, let the forces be parallel and let C be a point on the line of resultant. From C draw $CD = p$, and $CE = q$, perpendiculars on the lines of P and Q respectively. Then by Arts. 67 and 69

$$\frac{CD}{CE} = \frac{Q}{P}$$

and the moments are evidently of opposite signs,

therefore $$Pp + Qq = 0.$$

Conversely it may be shown that *if the algebraic sum of the moments of any two forces with respect to any point is equal to zero, the point must be on the line of action of resultant.*

86. PHYSICAL MEANING OF A MOMENT.—When two forces acting on a rigid body have a resultant, the resultant is in every way equivalent to the forces, and they may be replaced by it. Hence, when any point on the line of action of the resultant, rigidly connected with the body, is fixed, the tendency to produce motion is counteracted, and the two forces are in equilibrium with the resistance of the fixed point.

It is evident that the tendencies of the forces P and Q would be to turn the body round an axis through C, perpendicular to their plane, in opposite directions. Fig. Art. 85. These tendencies to cause rotation balance when the point C is any point on the line of resultant, since the forces might be replaced by a single force R through C. We have shown that when this is the case the moments of the forces P and Q with respect to C are equal and opposite. Hence, *the moment of a force with respect to a point is the mathematical measure of its power in producing, or tending to produce, rotation, in a body rigidly connected with the point, round an axis through the point perpendicular to the plane of the point and the force.*

The effect of a force in producing rotation round a point is thus directly proportional to the magnitude of the force, and also directly proportional to the length of the perpendicular drawn from the point on the line of the force.

87. PRINCIPLE OF THE LEVER.—A *lever* is a rigid body which is capable of turning freely in one plane about a fixed point, or fixed axis, which is called the *fulcrum.*

When two forces act on a lever in the plane in which it is capable of turning, the lever is in equilibrium when the moments of the forces round the fulcrum are equal and opposite.

This follows from Art. 85, and is called the *principle of the lever*. The resultant of the forces passes through the fulcrum, by the resistance of which it is counteracted.

The reaction of the fulcrum is equal in magnitude and opposite in direction to the resultant of the two forces. Art. 17.

The *principle of the lever* has been deduced from the *parallelogram of forces;* this principle is however sometimes taken as the fundamental one in Statics and the *parallelogram of forces* deduced from it.

88. *Assuming the Principle of the Lever to deduce the Parallelogram of Forces as far as the line of action of the resultant is concerned.*

Let P and Q be two forces acting on a rigid body, and meeting at O. Fig. Art. 85. Their resultant lies in their plane and is intermediate in position.

Suppose the line of action of the resultant to be OC; the forces may be replaced by a single force along OC.

Hence if any point on OC such as C were fixed, the resultant would be counteracted by the resistance of the fixed point; therefore the forces P and Q would balance round a fulcrum C.

We have by the *principle of lever*, without regarding the signs of moments,

$$P.\,CD = Q.\,CE$$

or

$$\frac{P}{Q} = \frac{CE}{CD}.$$

But by similar triangles CAD and CBE

$$\frac{CE}{CD} = \frac{CB}{CA} = \frac{OA}{OB}$$

Therefore

$$\frac{P}{Q} = \frac{OA}{OB}$$

Therefore OA and OB represent the forces, and the diagonal of the parallelogram $OACB$ is the line of action of their resultant. Having shown that the diagonal is the line of action of the resultant, we can prove as in Art. 35

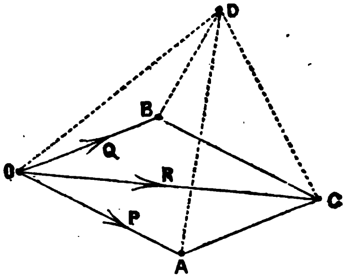
that it also represents the magnitude of resultant, and hence the *parallelogram of forces* can be completely demonstrated.

89. *The algebraic sum of the moments of two forces, which have a single resultant, round any point in the plane containing the forces is equal to the moment of their resultant.*

This follows from the physical meaning of a moment, since the resultant must be in every way equivalent to the forces, as well in tending to produce rotation round any point, supposed to be rigidly connected with the body on which the forces act, as in tending to produce motion along its line of action.

The following proof depending on the definition of a moment and the *parallelogram of forces* is usually given in the case when the lines of action of the forces meet at a point.

Let P and Q be two forces meeting at O. Take OA and OB to represent the forces and complete the parallelogram; OC represents their resultant R.



Let D be any point in the plane of forces; join D to O, A, C, B; we shall show that, taking D as a common vertex, the area of the triangle on resultant is equal to the sum of the areas of triangles on components.

For, area DOA = area DBC + area OBC; because the base OA of DOA is equal to the common base BC of DBC and OBC and the altitude of DOA is evidently equal to the sum of the altitudes of DBC and OBC.

Adding area DOB to both sides, we have

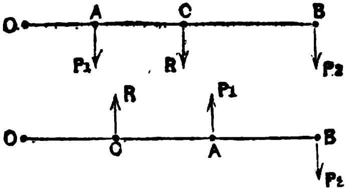
$$\text{area } DOA + \text{area } DOB$$
$$= \text{area } DBC + \text{area } OBC + \text{area } DOB = \text{area } DOC.$$

Therefore, 2area DOA + 2area DOB = 2area DOC

$\therefore$ moment of P + moment of Q = moment of R.

When the point D lies within the angle AOB the moments of the forces P and Q will have opposite signs, the moment of the resultant being of the same sign as the greater. In this case the *algebraic* sum of the moments of P and Q is still equal to the moment of the resultant R. The student should draw a diagram to represent this case and *write* out the proof.

Cor.—The proposition of this article can be easily extended to any number of forces meeting at a point. For the sum of the moments of any two of the forces is equal to the moment of their resultant, and this resultant can be compounded with another force, and so on. Hence the moment of the resultant is equal to the algebraic sum of the moments of the components.



90. It remains to prove the proposition for two parallel forces.

Let P_1 and P_2 be two parallel forces and R their resultant.

From any point O in their plane draw a line perpendicular to their lines of action meeting them in A, B, C.

$$\text{Let } OA=x_1;\quad OB=x_2;\quad OC=a.$$

Then by Arts. 67 and 69, from the first figure,

$$\frac{P_1}{P_2}=\frac{CB}{CA}=\frac{x_2-a}{a-x_1};\quad P_1+P_2=R.$$

$$\therefore\quad (P_1+P_2)\,a=Ra=P_1\,x_1+P_2\,x_2$$

Moment of P_1+moment of P_2 = moment of R.

In the second figure, when the forces act in opposite directions, we get

$$\frac{P_1}{P_2}=\frac{CB}{CA}=\frac{x_2-a}{x_1-a};\quad P_1+P_2=R.$$

$$(P_1-P_2)\,a=Ra=P_1\,x_1-P_2\,x_2$$

and since the moment $P_2\,x_2$ is negative we have the same result.

The student will observe that the proposition of this article has been before proved for any number of parallel forces lying in the same plane. Art. 81, case III. For, when the line Ox is drawn perpendicular to the forces, $P_1\,x_1$, $P_2\,x_2$, ... are the moments of the forces about O, and $\bar{x}\,\Sigma P$ is the moment of their resultant.

When the point round which the moments are taken lies on the resultant, the moment of the resultant vanishes, and hence the sum of the moments of the forces vanishes which is the proposition of Art. 85.

91. There is one case when the sum of the moments of the two forces is never zero, namely when the forces are equal, parallel and tend in opposite directions. Such a pair of forces is called a couple, Art. 74, and does not admit of a single resultant.

The algebraic sum of the moments of the two forces which constitute a couple, round any point in their plane, is constant and equal to the magnitude of either force multiplied by the perpendicular distance between them.

Let P_1 and P_2 be equal parallel forces tending in opposite directions, second figure, Art. 90; let O be any point in their plane.

$$\text{Moment of } P_2 = P_2\,OB = P_1\,OB.$$

$$\text{Moment of } P_1 = -P_1\,OA.$$

$$\text{Moment of } P_1 + \text{moment of } P_2 = P_1\,(OB - OA).$$

$$= P_1\,AB = \text{a constant.}$$

When the point O lies between the lines of action of the equal forces, their moments about O have the same sign, as both forces then tend to turn in the same direction round O; we thus get the same result.

92. When a system of forces lying in a plane, and either meeting at a point or parallel to each other, is in equilibrium, any one of them is equal in magnitude and opposite in direction to the resultant of all the others; hence the moment of any one about any point in their plane must be equal and opposite to the sum of the moments of the rest

(Arts. 89, 90); hence, *when such a system of forces is in equilibrium, the algebraic sum of the moments of the forces round any point in their plane is zero.*

This also follows from the nature of a moment and expresses the fact that the forces produce no tendency to rotation round the point.

The above, although a *necessary* condition for equilibrium, is not a *sufficient* condition; as the forces might have a resultant passing through the point round which the moments are taken, in which case the algebraic sum of the moments of the forces would be zero, but they would not be in equilibrium.

It will be easily seen that the following conditions of equilibrium are both necessary and sufficient.

A system of forces lying in a plane and meeting at a point will be in equilibrium when the algebraic sum of the moments of the forces vanishes round any two points, in the plane of the forces, which are not situated on a straight line passing through the point at which the forces meet.

A system of parallel forces in one plane will be in equilibrium when the algebraic sum of the moments of the forces vanishes round any two points, in the plane of the forces, which are not situated in a straight line parallel to the forces.

93. THE MOMENT OF A FORCE WITH RESPECT TO A LINE.— Resolve the force into two components, one parallel, the other perpendicular to the line; the product of the component perpendicular to the line and the shortest distance between the line and the direction of this component is the moment of the force with respect to the line.

The physical meaning of the moment of a force with respect to a line may be thus explained. Let A and B be two points fixed in a rigid body which is acted upon by a force R, not in the same plane with the line AB. The only motion the body can have is a motion of rotation round the axis AB. Let R be resolved into two components— one P parallel to the line AB, the other Q perpendicular to AB. Then the effect of R on the body is the same as the combined effect of P and Q. But P being in the same plane with AB can evidently produce no motion in the body, since AB is fixed. Hence the effect of R in producing rotation round AB is the same as the effect of Q. Through Q draw a plane perpendicular to AB meeting it in C, and let CD be the perpendicular from C on Q. Then $Q.CD$ measures the effect of Q in producing rotation round AB and is the moment of R with respect to AB.

Obs.—The moment of a force with respect to a line can only vanish when the line and the direction of the force are in the same plane. This includes the cases when the line and force are parallel and when the line intersects the direction of the force.

94. *When three forces maintain a body in equilibrium they must lie in the same plane.*

Suppose a body to be in equilibrium under the action of three forces P, Q, R. Take on the lines of action of P and Q any points A and B, respectively, so that AB shall not be parallel to the line of action of R. We may suppose these points to become fixed without disturbing the equilibrium. Art. 16.

The moments of P and Q round AB are separtely zero. Hence, for equilibrium, the moment of R with respect to the line AB must be zero, otherwise R would produce a motion of rotation in the body round AB. Therefore, since AB is not parallel to R, the line of action of R must intersect AB.

Since then any straight line which meets the lines of action of P and Q and is not parallel to R meets the line of action of R, the lines of action of P, Q, R must lie in one plane.

95. We can now give a complete account of the conditions of equilibrium of three forces acting on a rigid body. See Arts. 64 and 72.

(1.) *The lines of action of the forces must lie in the same plane.*

(2.) *They must meet at a point or be parallel.*

(3.) *Any one of the forces must be equal and opposite to the resultant of the other two.*

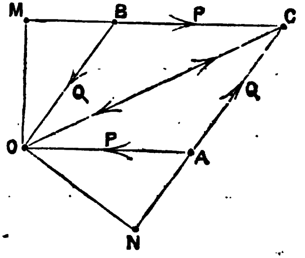
96. *Def.*—The *arm* of a couple is the perpendicular distance between the lines of action of its forces.

The *moment of a couple* is the product of one of the equal forces and the arm.

The tendency of a couple acting on a rigid body is evidently to cause rotation, and the power of the couple to cause rotation round every point in its plane is the same, and is measured by the moment of the couple.

Couples which tend to turn a body round in the same way may be called *like*, and couples which tend to turn a body round in opposite ways may be called *unlike;* their moments have opposite signs.

97. *Two unlike couples in the same plane equilibrate when their moments are equal.*



Let the equal forces of one couple be P; of the other Q. The lines of action will in general form a parallelogram; let it be $OACB$. Draw OM perpendicular to BC, and ON perpendicular to AC.

Then the moments of the couples are $P.OM$ and $Q.ON$. Hence, by hypothesis,

$$P.OM = Q.ON$$

therefore
$$\frac{P}{Q} = \frac{ON}{OM} = \frac{OA}{OB}$$

Hence OA and OB represent P and Q; and, by the Parallelogram of Forces, CO represents the magnitude and direction of their resultant.

Similarly it may be shown that the resultant of P and Q at C is represented in magnitude and direction by OC; hence the two resultants equilibrate.

Cor. 1.—Since two unlike couples of equal moments in the same plane balance each other, *two like couples in the same plane produce equal effects when their moments are equal.*

Cor. 2.—Since a couple is always balanced by an unlike couple of equal moment in the same plane, it follows that *the effect of a couple will not be altered if we change it into* *another like couple of equal moment, or if we transfer it to any position in its own plane.*

*Obs.*1.—In the proof of this article it is tacitly assumed that the forces of one couple are not parallel to those of the other. This restriction may now be removed since we see that the effect of any couple is not altered by turning it into a new position in its own plane. The student will, however, have no difficulty in proving the proposition directly when the four forces are parallel.

Obs. 2.— The proposition of this article follows at once from the nature of a couple. For the tendency of a couple is to produce rotation and the measure of the power of the couple is its moment (Art. 96); hence if two unlike couples in the same plane have equal moments they must balance.

98. *To find the resultant of any number of couples acting on a rigid body in the same plane.*

Let $P, Q, R,$ ………… be the forces of couples:

$a, b, c,$ ………… their arms.

Let the couples be transferred so as to have their arms in the same straight line, and one extremity common. This will not alter their effects. Art. 97.

Let them be now replaced by other couples having equal arms, their moments remaining unchanged. This will not alter their effects. Art. 97.

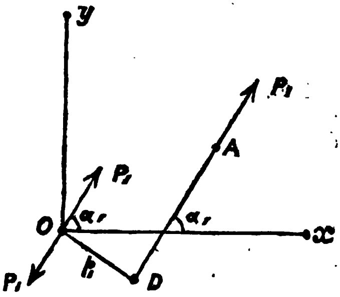
Let the common arm be a and the new forces of couples $P', Q', R',$ ………; we have $P'a = Pa$; $Q'a = Qb$; $R'a = Rc$, ………

The forces $P', Q',$ ………… acting in the same line at one extremity of the common arm, may be replaced by their resultant equal to their algebraic sum; and likewise the equal and parallel forces at the other extremity of common arm may evidently be replaced by an equal and parallel resultant.

The force of the resultant couple is $P + Q' +$ ………. and its arm a; hence its moment is $(P' + Q' + R' + \ldots\ldots\ldots\ldots)\,a = Pa + Qb + Rc +$ ……………

Hence *the moment of the resultant couple is equal to the algebraic sum of the moments of the original couples.*

99. *To determine the resultants of any number of forces acting on a rigid body in one plane, and the conditions of equilibrium.*



Take any two axes Ox and Oy at right angles in the plane of the forces.

Let P_1 be one of the forces, acting at A.

Introduce at O two forces parallel and equal to P_1 and acting in opposite directions; they will not affect the system.

From O draw $OD = p_1$ perpendicular to the line of action of P_1.

The force P_1 is thus equivalent to an equal, parallel and like force at O, and a couple whose moment is $P_1 p_1$.

Let α_1 be the angle which the line of action of P_1 makes with Ox. P_1 at O is equivalent to $P_1 \cos \alpha_1$ along Ox, and $P_1 \sin \alpha_1$ along Oy.

The same process may be applied to all the forces. We would thus get two forces, $\Sigma P \cos \alpha$ along Ox and $\Sigma P \sin \alpha$ along Oy, and a system of couples whose moments are $P_1 p_1$, $P_2 p_2$..................

Hence the system may be replaced by a single resultant force R, and a single resultant couple G, where

$$R^2 = (\Sigma P \cos \alpha)^2 + (\Sigma P \sin \alpha)^2$$

$$G = \Sigma P p.$$

The forces must be in equilibrium when $R = 0$ and $G = 0$.

Therefore the necessary, and *sufficient*, conditions of equilibrium are three in number:

$$\Sigma P \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\Sigma P \sin \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\Sigma P p = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The moment of the resultant couple is equal to the sum of the moments of the forces round any point O in their plane; hence the above equations of condition may be expressed in words—

The algebraic sum of the resolved parts of the forces parallel to any two directions at right angles must be separately zero, and the algebraic sum of the moments of the forces round any point in their plane must be zero.

100. REMARKS ON CONDITIONS OF EQUILIBRIUM.—If the sums of the resolved parts of the forces parallel to *any* two directions are separately zero, it can be shown, as in Art. 60, that the sums of the resolved parts of the forces parallel to two rectangular directions must be also separately zero, and therefore the conditions (1) and (2) of the last article are satisfied. Hence the directions of resolution need not be rectangular.

The conditions of equilibrium of any number of forces in one plane may be expressed in various forms, but it will be found in general that these conditions give rise to three, and only three, independent equations of equilibrium.

For example, as in Art. 60, we see that the equation (2) follows from (1) and the equation got by resolving the forces in any other direction. Hence no new independent equation can be got by resolving the forces in more directions than two.

It will be also found that we can get no new equations independent of (1), (2) and (3) by taking moments round any other point.

In solving problems the equations of equilibrium are obtained in the most convenient forms by a judicious choice of the directions of resolution, and of the point round which the moments are taken. We have the following rules for forming them:

(1.) *Resolve all the forces parallel to any direction, and equate their algebraic sum to zero.*

(2.) *Resolve all the forces parallel to any other direction, and equate their algebraic sum to zero.*

(3.) *Take moments of all the forces round any point, and equate their algebraic sum to zero.*

The student should consider carefully the physical meanings of these equations of equilibrium. The first two express that the combined action of the forces has no tendency to impart a motion of translation to the body in any direction.

The third expresses that the combined action of the forces has no tendency to impart a motion of rotation to the body, and hence the body must be at rest.

101. *A system of forces, not in equilibrium, acting in one plane on a rigid body can be reduced either to a single couple in the plane of the forces or to a single force.*

For it has been shown (Art. 99) that the forces can in general be reduced to a single force R acting at any arbitrary point O in their plane and a couple G.

If $R=0$, the forces reduce to a single couple G. If $G=0$, the forces reduce to a single force R. When neither R nor G vanishes, the system may be reduced to a single force. For, one of the equal forces of G may be supposed to act through O, and not in the same straight line with R. Art. 97. The resultant of R, and this force acting at O, cannot be parallel to the other force of the couple G, and hence can be compounded with it so as to give a single final resultant.

102. *A system of forces acting in one plane on a rigid body will be in equilibrium if the algebraic sum of the moments of the forces vanishes round any three points in the plane, which are not in the same straight line.*

For, if the system be not in equilibrium it is equivalent to a single resultant or a couple. Art. 101.

But the system cannot be equivalent to a couple; for then the sum of the moments could not vanish round any point in the plane of forces. Art. 91.

Also the forces of the system cannot have a single resultant; for then the sum of the moments could only vanish for points on the line of the resultant. Hence the system must be in equilibrium.

EXAMPLES.

1. Three forces, P, Q, R, act through the angular points of a triangle in directions, perpendicular to the opposite sides; find the conditions of their equilibrium, by taking moments round the angular points.

Result. $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$.

2. Three forces act perpendicularly to the sides of a triangle at the middle points, and each force is proportional to the side at which it acts; show, by taking moments round the angular points of the triangle, that the forces equilibrate.

3. ABC is a triangle; D, E, F are the middle points of the sides opposite to A, B, C, respectively; show, by taking moments round the angular points, that forces represented by AD, BE and CF are in equilibrium.

4. Q and P are fixed points on the circumference of a circle; QA and QB are any pair of chords drawn at right angles through Q; if QA and QB represent forces, show that the algebraic sum of their moments with respect to P is constant.

5. Forces act along the sides of a triangle; show that they may be reduced to a single resultant passing through the centre of the inscribed circle, when one of the forces is equal to the sum of the other two and tends to produce rotation in the opposite direction.

6. Forces P, Q, R act along the sides BC, CA, AB of a triangle, and their resultant passes through the centres of the inscribed and circumscribed circles; shew that

$$P : Q : R :: \cos B - \cos C : \cos C - \cos A : \cos A - \cos B.$$

7. $ABCD$ is a quadrilateral inscribed in a circle; forces P, Q, R act in directions AB, AD, CA; show that they will be in equilibrium when $P : Q : R :: CD : BC : BD$.

8. A number of forces in the same plane meet at a point, find the locus of a point such that the algebraic sum of the moments of the forces round the point shall be constant.

Result. A line parallel to resultant.

9. Forces P, Q, R, S act along the sides AB, BC, CD, DA of a parallelogram; find the position of their resultant.

Let $AB = a$, $AD = b$, and let x and y be the distances on AB and AD respectively, measured from A, at which the resultant cuts these lines.

Result. $$x = \frac{Qa + Rb}{Q - S},$$ $$y = \frac{Qa + Rb}{R - P}.$$

10. Forces are represented in magnitude and position by the sides of a plane polygon taken in order; show that their resultant is a couple whose magnitude is represented by twice the area of the polygon.

11. Forces are represented by the sides of a plane polygon, except one, taken in order; show that they are equivalent to a single force which is parallel to the omitted side.

CHAPTER V.

CENTRE OF GRAVITY.

103. Every particle of matter on the earth is acted on by a force proportional to the mass of the particle (Art. 8) the line of action of the force being vertical, and its point of application the point which determines the position of the material particle.

Def.—A horizontal plane at any point of the earth's surface is a plane parallel to the free surface of a liquid at rest. Any line parallel to this plane is called a horizontal line, and straight lines perpendicular to this plane are called vertical lines. A vertical line at any point is the line of action of the force of gravity at that point.

Obs.—Through a given point an infinite number of horizontal lines can be drawn, but only one vertical line.

104. *The action of gravity on a system of material particles, supposed to be rigidly connected, is reducible to a single resultant force whose line of action passes through a point, fixed relatively to the particles whatever be the position of the system. This point is called the Centre of Gravity of the particles.*

Let the masses of the particles be m_1, m_2, m_3,.........; their positions a, b, c,..........

The forces with which the particles gravitate (the weights of the particles) are m_1g, m_2g, m_3g,.........; their lines of action are all vertical, and hence parallel; and their points of application are a, b, c,..........

Hence, by the nature of like parallel forces, they are equivalent to a single resultant equal to their sum, and whose point of application is determined by the following construction.

Cut ab at k and kc at l, and so on; so that

$$\frac{ak}{bk}=\frac{m_2g}{m_1g}=\frac{m_2}{m_1};$$

$$\frac{kl}{cl}=\frac{m_3g}{(m_1+m_2)g}=\frac{m_3}{m_1+m_2};$$

……

Proceeding in this way we find a point G through which the resultant force of gravity on the system, $g\,\Sigma\,(m)$, acts.

The position of G thus depends only on the masses of the particles and the distances between them, and is therefore fixed relatively to the particles whatever be the position of the system.

The point G is called the Centre of Gravity and is the same as the centre of a system of like parallel forces, the weights of the particles (which are proportional to their masses), whose points of application are the positions of the particles. We have here merely repeated the process of Art. 77 for the sake of clearness.

Obs.—In the above construction we are only concerned with the ratios of the weights, which are the same as the ratios of the masses of the particles. Hence we may use, indifferently, the numerical expressions for the weights referred to any the same unit of weight, or the numerical expressions for the masses referred to any the same unit of mass.

105. CENTRE OF GRAVITY OF A RIGID BODY.—A rigid body may be considered as made up of an assemblage of material particles rigidly connected together, the mass of the body being equal to the sum of the masses of the particles which compose it. Hence, by the last article, the action of gravity on the body is reducible to a single resultant force, the weight of the body, whose line of action passes through a fixed point in the body, whatever be its position.

Def.—The Centre of Gravity of a rigid body is a definite point in the body through which the line of action of the resultant force of gravity always passes, whatever be the body's position.

In this definition the term body may be understood to include a system of particles or a system of bodies rigidly connected, or supposed to be rigidly connected.

The construction of Art. 104 shows that the effect of gravity on any body is the same as if the whole mass of the body were collected at the centre of gravity.

Obs.—The point determined by the construction of Art. 104 depends only on the masses of the particles and their distances. Considering it from this point of view it is called the *Centre of Inertia* of the masses, and every body or system of bodies has got a centre of inertia. The general theory of the resultant action of gravity shows that there is not always one point which can properly be called the centre of gravity of a rigid body. In the case of terrestrial gravitation the gravitating forces of the particles which compose the same body may be all considered parallel and proportional to the masses of the particles, and hence have a resultant force which always passes through the centre of inertia of the body. As this is the only case we shall have to consider, we may, according to common custom, use the term "Centre of Gravity" as equivalent to "Centre of Inertia." The ideas involved in the terms are, however, essentially different.

106. *A body or a system of bodies cannot have more than one centre of gravity.*

This follows from the nature of like parallel forces, or may be at once deduced from the definition. For, if possible, suppose there are two centres of gravity, G and G'. Let the body be placed so as to have the line GG' horizontal, then, by the definition of the centre of gravity, the line of action of the resultant force of gravity must pass through G, and also G'; but this is impossible, since the line of action must be vertical, while GG' is horizontal.

107. *If the centre of gravity of a body, acted on only by gravity, be fixed or supported, the body will rest in any position.*

For in every position the resultant force of gravity on the body acts vertically through the fixed point, and will hence be equilibrated by its equal and opposite reaction. Art. 17.

The effect of gravity on the body is in fact the same as if the whole mass were collected at the centre of gravity, hence the body will be at rest when the centre of gravity is supported. In the case of a system of bodies their common centre of gravity must be supposed to be rigidly connected with the bodies of system.

This article suggests another definition of the centre of gravity, which is very commonly used.

Def.—The centre of gravity of a rigid body is a point on which the body will balance in all positions, supposing the point to be supported, and the body acted on only by gravity.

108. *A body which can turn freely round a fixed point, or which is suspended from a fixed point, will not rest unless its centre of gravity be in the vertical line passing through the fixed point.*

For the body is acted on only by two forces, the force of gravity which acts vertically through the centre of gravity, and the reaction of the fixed point. Hence the body can be in equilibrium only when these forces have the same line of action; and when this is the case the body must be in equilibrium; for the fixed point will supply a reaction equal and opposite to the force of gravity. Art. 17.

Hence the body is, and only is, in equilibrium when the vertical line through the centre of gravity passes through the fixed point.

109. *To determine the position of the centre of gravity of any body experimentally.*

Let the body be suspended from a fixed point in any way: when it has arrived at a state of rest the vertical line through the fixed point must, by last article, pass through the centre of gravity; let the direction of this line in the body be determined. Again suspend the body from another different point; we shall obtain a second line passing through the centre of gravity; consequently the intersection of these two lines will be the centre of gravity.

110. When a body is *slightly* displaced from a position of equilibrium, it may (1) tend to return to its original

position, (2) tend to deviate further from the original position, (3) remain in equilibrium in the new position.

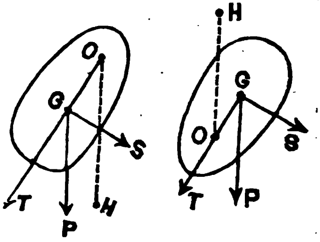
In (1), the original position of equilibrium is called *stable*; in (2), *unstable*; in (3), *neutral.*

111. *When a body, which can turn freely round a fixed point, is in equilibrium, the equilibrium is stable, unstable or neutral according as the centre of gravity is vertically below, vertically above, or coincident with, the fixed point, the body being only acted on by gravity and the reaction of the fixed point.*

Let O be the fixed point, G the centre of gravity of body, OH a vertical line through O.

When G is on OH below O, the body is in stable equilibrium.

For, suppose the body to be slightly displaced from the position of equilibrium, as in the first figure. Let the force of gravity, P, acting vertically through G, be resolved into two components, T along OG, and S perpendicular to OG.



T is counteracted by the reaction of the fixed point; S is unresisted, and tends to turn the body round to its original position of equilibrium with G vertically below O.

Similarly it may be shown, from the second figure, that the position of equilibrium, when G is vertically above O on OH, is unstable.

When O coincides with G, the body will rest in any position (Art. 107), therefore the equilibrium is neutral.

112. *When a body is placed on a horizontal plane, or on an inclined plane so rough as to prevent slipping, it will stand or overturn according as the vertical line through its centre of gravity falls within or without the base, the body being supposed to be acted on only by gravity and the reactions of the different points of the plane with which it is in contact.*

By base is here to be understood the area enclosed by stretching a string round the points of contact of the body with the plane.

For, since the reactions of the points of contact of the body with the plane all act from the plane, the line of action of their resultant cannot cut the plane outside the base of the body.

This resultant will adjust itself so as to counteract the force of gravity on the body (Art. 17) as long as the vertical line through the centre of gravity falls inside the base.

But when the verticle through the centre of gravity falls outside the base, the resultant reaction can no longer have the same line of action with the resultant force of gravity on the body, and hence there cannot be equilibrium.

113. *To determine analytically the centre of gravity of any number of particles whose positions are given.*

Let the masses of the particles be m_1, m_2, m_3 .... ; and let a, b, c, ... be their positions.

The weights of the particles are m_1g, m_2g, m_3g .... and are all vertical, and therefore parallel forces. The centre of this system of parallel forces is the centre of gravity.

Hence if the co-ordinates of a, b, c, ... with respect to any three planes be $(x_1, y_1, z_1,)$ (x_2, y_2, z_2) (x_3, y_3, z_3) .... the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of gravity are determined, as in Art. 79, to be

$$\bar{x}=\frac{\Sigma mx}{\Sigma m};\quad \bar{y}=\frac{\Sigma my}{\Sigma m};\quad \bar{z}=\frac{\Sigma mz}{\Sigma m}.$$

I. When the particles lie in the same plane, and the co-ordinates of the particles with respect to two axes in the plane are given, the co-ordinates of the centre of gravity are found from—

$$\bar{x}=\frac{\Sigma mx}{\Sigma m};\quad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

II. When the particles lie on a line, and their distances from any point on the line are given, the distance of their centre of gravity is found from—

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}.$$

117. *To find the centre of gravity of a uniform material line.*

The material line may be supposed to be made up of an indefinite number of equal material particles, which may be considered in pairs equidistant from the middle point of the line.

The centre of gravity of each pair of equal particles, equidistant from the middle point of the line, is that middle point.

Hence, the common centre of gravity of the pairs of particles, that is, the centre of gravity of the line, is its middle point.

118. A plane lamina of any form may be supposed to be made up of an indefinite number of parallel material lines; the centres of gravity of these lines are their middle points.

If these middle points all lie on a line, that line is called in geometry a *rectilinear diameter*, and the common centre of gravity of all the material lines, that is, the centre of gravity of the lamina, must lie on this diameter.

Hence, if we can draw two rectilinear diameters in any plane figure, the centre of gravity of the figure must lie on both, and is, therefore, their point of intersection.

This principle is often of use in finding the centres of gravity of plane figures. We give some examples.

(1.) The centre of gravity of a circular lamina is its centre.

(2.) The diagonals of a parallelogram are diameters, hence its centre of gravity is their intersection.

(3.) The lines joining the middle points of the sides of a triangle to the opposite vertices are diameters, hence its centre of gravity is their intersection.

(4.) The line joining the middle points of the parallel sides of a trapezium is a diameter, and hence its centre of gravity is on that line.

119. A body is said to be *symmetrical* with respect to a plane when the body may be supposed to be made up of pairs of equal particles on opposite sides of the plane,

equidistant from it and on the same perpendicular to it. Similarly, a body may be symmetrical with respect to a line or a point.

When a body is symmetrical with respect to a plane, a line, or a point, the centre of gravity lies in the plane, on the line, or coincides with the point. For the centre of gravity of each pair of equal particles must be in the plane, on the line, or at the point.

The principle of symmetry is often of use in finding the centres of gravity of bodies. We give some examples.

(1.) The centre of gravity of a solid sphere, or the centre of gravity of the surface of a sphere, is its centre.

(2.) The centre of gravity of a cube is its geometrical centre.

(3.) The centre of gravity of a solid cylinder and the centre of gravity of the surface of a cylinder are at the middle point of its axis.

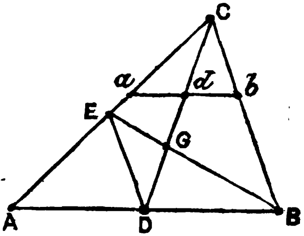
(4.) The centre of gravity of a solid right circular cone and the centre of gravity of the surface of the cone lie on its axis.

The following examples of the determination of the centre of gravity are usually given.

120. *To find the centre of gravity of a triangle (triangular lamina).*

Let ABC be the triangular lamina; it may be supposed to be made up of an indefinite number of material lines parallel to the base (Art. 116); let ab be one of these lines.

Join C to D, the middle point of AB, meeting ab in d; then by similar triangles



$$\frac{ad}{AD}=\frac{dC}{DC}=\frac{db}{DB}$$

But $AD=BD$; therefore $ad=db$. Hence d is the middle point, and therefore the centre of gravity of ab. Art. 117.

Similarly it may be shown that the centre of gravity of every material line parallel to AB lies on CD; hence the centre of gravity of the triangle lies on CD.

Join B to E the middle point of AC; in the same way it may be shown that the centre of gravity of the triangle lies on BE.

Hence the centre of gravity of the triangle is G, the point of intersection of CD and BE.

Join DE; then DE is parallel to BC (Euclid, VI, 2); therefore by similar triangles

$$\frac{DG}{GC}=\frac{DE}{BC}=\frac{AD}{AB}=\frac{1}{2}$$

Thus GC is twice DG, and therefore DG is one third of DC; similarly EG is one third of EB.

Hence the centre of gravity of a triangle is determined by the following rule. Join any vertex to the middle point of the opposite side; the centre of gravity is on this line at one third of its length from the side.

Cor. 1.—It follows by similar triangles that the perpendicular distance of the centre of gravity from the base is one-third of the altitude of the triangle.

Cor. 2.—The centre of gravity of a polygon can be found by dividing it into triangles.

Obs.—The position of the centre of gravity of a triangle was indicated in Art. 118 (3), but it is usual to give, as above, the proof of the geometrical proposition that the lines joining the middle points of the sides to the opposite vertices are diameters, meet at a point, and trisect each other.

121. Geometrical propositions can sometimes be proved by the aid of the principle that a body can have only one centre of gravity. Thus all rectilinear diameters of a plane figure must meet at a point, because they must all pass through its centre of gravity.

The lines joining the middle points of the sides of a triangle to the opposite vertices meet at a point.

The diagonals of a parallelogram and the lines joining the middle points of the opposite sides, being all diameters, meet at a point.

122. *The centre of gravity of a triangle coincides with the centre of gravity of three particles of equal masses placed at the angular points.*

Let ABC be the triangle (Fig. Art. 120). Bisect AB at D; then D is the centre of gravity of equal particles at A and B.

Join DC and cut it at G so that DG may be to GC as 1 is to 2; G is the centre of gravity of the three equal particles at A, B and C, and G is also by the construction the centre of gravity of the triangle.

Cor.—Hence, if x_1, x_2, x_3, be the co-ordinates of the angular points of a triangle, the co-ordinate of the centre of gravity is (Art. 113)—

$$\bar{x}=\frac{mx_1+mx_2+mx_3}{m+m+m}=\frac{x_1+x_2+x_3}{3}.$$

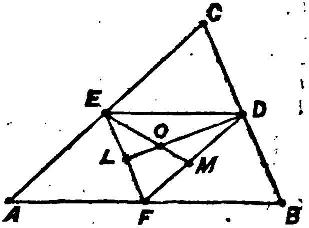
123. *The centre of gravity of the perimeter of a triangle coincides with the centre of the circle inscribed in the triangle formed by joining the middle points of the sides.*

The sides are supposed to be material lines of uniform section and density; hence their masses are proportional to their lengths.

Let ABC be the triangle; the middle points D, E, F, of the sides are their centres of gravity.

Join EF and cut it at L so that EL may be to FL as AB is to AC; then L is the centre of gravity of the sides AC and AB.

Join DL, the common centre of gravity of the three sides lies on DL; and we have—

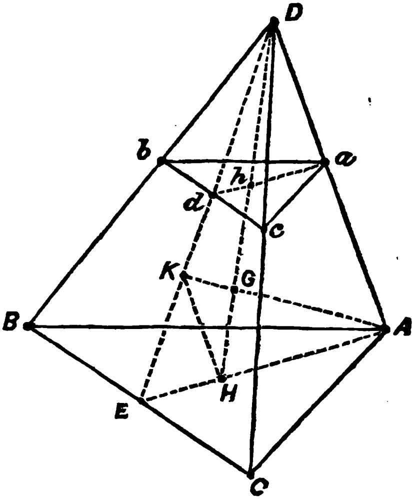


$$\frac{EL}{FL}=\frac{AB}{AC}=\frac{ED}{FD}, \text{ by similar triangles;}$$

therefore LD bisects the angle EDF (Euclid VI, 3), and therefore the centre of the circle inscribed in DEF lies on LD (Euclid IV, 4).

Join FD and cut it at M so that FM may be to DM as BC is to AB. It may be shown in a similar manner that the centre of gravity of the three sides, and also the centre of the circle inscribed in DEF, lies on EM.

Hence O, the point of intersection of EM and DL, is the centre of gravity of the perimeter of the triangle ABC, and also the centre of the circle inscribed in the triangle DEF.



124. *To find the centre of gravity of a triangular pyramid.*

Let $ABCD$ be the pyramid; it may be supposed to be made up of an indefinite number of triangular laminas parallel to the base ABC; let abc be one of these triangles.

Bisect BC in E, and take, on EA, EH one-third of EA; H is the centre of gravity of the triangle ABC.

Join ED meeting bc in d, and join HD meeting da in h.

Then by similar triangles, $bd = dc$

and
$$\frac{dh}{EH} = \frac{Dh}{DH} = \frac{ah}{AH}$$

But AH is twice EH; therefore ah is twice dh; hence h is the centre of gravity of triangle abc.

Similarly it may be shown that the centres of gravity of all the triangular laminas, parallel to ABC, lie on HD. Hence the centre of gravity of the pyramid lies on HD.

Again, take EK one-third of ED; K is the centre of gravity of the triangle BCD, and similarly it may be shown that the centre of gravity of the pyramid lies on AK.

Hence the centre of gravity of the pyramid must be G the point of intersection of HD and AK.

Join HK, then HK is parallel to DA (Euclid, VI, 2) and by similar triangles,

$$\frac{HG}{GD}=\frac{HK}{DA}=\frac{EH}{EA}=\frac{1}{3}$$

Thus HG is one-third of DG, that is, HG is one-fourth of HD, and similarly KG is one-fourth of KA.

Hence the centre of gravity of a triangular pyramid is determined by the following rule. Join any angular point with the centre of gravity of the opposite face; the centre of gravity of the pyramid is on this straight line, at one-fourth of its length from the face.

Cor.—The lines joining the angular points of a pyramid to the centres of gravity of the opposite faces meet at a point, the centre of gravity of pyramid.

125. *The centre of gravity of a triangular pyramid coincides with the centre of gravity of four particles of equal masses placed at its angular points.*

Let $ABCD$ be the pyramid (Fig. Art. 124). Take H the centre of gravity of triangle ABC; it is also the centre of gravity of equal masses at A, B and C. Art. 122.

Join HD and cut it at G so that HG may be to GD as 1 is to 3; G is the centre of gravity of four equal masses at A, B, C and D, and G is also, by the construction, the centre of gravity of the pyramid $ABCD$.

Cor. 1.—Hence if x_1, x_2, x_3, x_4 be the co-ordinates of the angular points of a pyramid, the co-ordinate of the centre of gravity is (Art. 113)—

$$x=\frac{x_1+x_2+x_3+x_4}{4}.$$

Cor. 2.—It follows from the last article, by similar triangles, or from the above value of the co-ordinate of the centre of gravity, that the distance of the centre of gravity from any face is one-fourth of the distance of the opposite vertex from the same face.

126. *To find the centre of gravity of a pyramid whose base is a plane rectilinear polygon.*

The pyramid may be supposed to be made up of an indefinite number of polygonal laminas parallel and similar to the base, and it may be shown, as in Art. 124, that the centre of gravity of each poligonal lamina lies on the line joining the vertex of pyramid to the centre of gravity of the base. Hence the centre of gravity of pyramid lies on on this line.

Again, by joining the centre of gravity of base to the angular points of the base and the vertex, the pyramid may be divided into triangular pyramids. The centre of gravity of every one of these lies on a plane parallel to the base at one-fourth of the distance of the common vertex from it. Hence the centre of gravity of the whole pyramid lies in this plane.

Therefore the centre of gravity of the pyramid is on the line joining the centre of gravity of base to the vertex, and at one-fourth of its length from the base.

127. *To find the centre of gravity of a solid cone.*

The reasoning of the last article will not be affected by supposing the number of sides of the polygonal base to be indefinitely great. In this case the polygon becomes a closed curve and the pyramid a cone.

Therefore the centre of gravity of a cone is on the line joining the vertex to the centre of gravity of base and at one-fourth of the length of this line from the base.

In the particular case of a right cone, the centre of the circular base is the centre of gravity of the base; hence *the centre of gravity of a right solid cone is on the axis at one-fourth of its length from the base.*

128. *The centre of gravity of the surface of a right cone is on the axis at one-third of its length from base.*

Suppose the circular base to be a regular polygon, the number of sides being indefinitely great and their magnitudes indefinitely small. Hence we may suppose the surface of the cone to be made up of an indefinitely great number of triangles whose bases are the sides of the polygon, and common vertex the vertex of the cone.

The centre of gravity of every triangle is at a distance from the base of the cone one-third of the distance of the vertex from the base. Art. 120. Hence the centre of gravity of the surface of the cone must lie in a plane parallel to the base and at a distance from it one-third of the distance of the vertex.

Again the surface of the cone is symmetrical with respect to its axis, hence the centre of gravity must lie on the axis (Art. 119) which proves the proposition.

Obs.—It may be easily shown that the centre of gravity of the surface of any cone is on the line joining the vertex to the centre of gravity of the perimeter of the base, at one-third of the length of this line from the base.

129. The masses of uniform bodies are proportional to the volumes of the bodies, and thus volumes may be substituted for masses in determining the centre of gravity.

The masses of uniform surfaces and laminas are proportional to their areas, and areas may be substituted for masses in determining the centre of gravity.

The masses of uniform material curves and lines are proportional to their lengths, and lengths may be substituted for masses in determining the centre of gravity.

Suppose a uniform surface or lamina to be made up of indefinitely small elements, the masses of these elements are proportional to the areas of the elements.

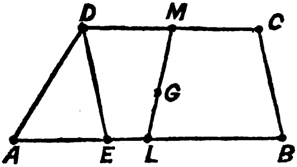
Let the areas of the elements be $a_1, a_2, a_3 \ldots\ldots$; and their co-ordinates $x_1, x_2, x_3 \ldots\ldots$.

The co-ordinates of the centre of gravity (Art. 113) become

$$\bar{x} = \frac{\Sigma ax}{\Sigma a}; \quad \bar{y} = \frac{\Sigma ay}{\Sigma a}; \quad \bar{z} = \frac{\Sigma az}{\Sigma a}.$$

These formulæ are used in Hydrostatics. We conclude this chapter with an example of their application in the solution of problems.

130. *To find the centre of gravity of a quadrilateral figure which has two sides parallel (a trapezium)*



Let the parallel sides be $AB = a$, and $DC = b$.

Draw DE parallel to BC; then $AE = a - b$; let the common altitude of the triangle ADE and the parallelogram $DEBC$ be p.

The perpendicular distances of the centres of gravity of the triangle and parallelogram from AB are respectively $\frac{p}{3}$ and $\frac{p}{2}$, and their areas, which are proportional to their masses, are $\frac{p}{2}(a-b)$ and pb respectively.

Let $\bar{x}$ be the perpendicular distance of the centre of gravity, G, of the quadrilateral from AB, then

$$\bar{x} = \frac{\frac{p}{2}(a-b)\frac{p}{3} + pb.\frac{p}{2}}{\frac{p}{2}(a-b) + pb} = \frac{p}{3} \cdot \frac{a+2b}{a+b}$$

Let $\bar{y}$ be the perpendicular distance of G from DC; similarly

$$\bar{y} = \frac{\frac{p}{2}(a-b)\frac{2p}{3} + pb.\frac{p}{2}}{\frac{p}{2}(a-b) + pb} = \frac{p}{3} \cdot \frac{b+2a}{a+b}$$

The line ML joining the middle points of the parallel sides is a diameter and contains the centre of gravity, Art. 118 (4), and

$$\frac{LG}{MG} = \frac{\bar{x}}{\bar{y}} = \frac{a+2b}{b+2a}$$

Hence the position of the centre of gravity is determined.

EXAMPLES.

1. Find the C.G. of three particles at the angular points of an equilateral triangle *ABC*, the masses of the particles at *B* and *C* being each double of the mass of particle at *A*.

Ans. From *A* draw *AD* perpendicular to *BC*; *G* lies on *AD* and $AD = 5\ DG$.

2. Two particles of equal masses are placed on two opposite sides of a parallelogram; show that their C.G. is fixed while they move along the sides so as always to be equidistant from opposite angles.

3. Particles are placed at the angular points of a triangle; if the masses of the particles be proportional to the lengths of the opposite sides, show that their C.G. is the centre ot the circle inscribed in the triangle.

4. If the base of a triangle and its area be given, show that the locus of its C.G. is a line parallel to the base.

5. A uniform rod, whose length is 3 feet and mass 4lbs., has a mass of 2lbs. placed at one end; find the centre of gravity of the system.

Ans. One foot from the end.

6. A uniform bar of iron 15 inches long, and whose mass is 12lbs., has a mass of 10lbs., suspended from one extremity; where must a fulcrum be placed that it may just balance upon it?

Ans. $4\frac{1}{11}$ in. from the end.

7. A uniform straght bar, whose weight is *W*, has masses whose weights are *P* and *Q*, suspended from its extremities; find the distance from the line of action of the weight *P*, of the fulcrum upon which the whole would balance, the length of the bar being *a*.

Ans. $\frac{a}{2}\cdot\frac{2Q+W}{P+Q+W}$.

8. A uniform triangular lamina is supported at its three angular points; show that, whatever be the form of the triangle, the pressures on the supports are all equal.

9. Four particles of 1, 2, 3 and 4 units of mass respectively are placed at equal intervals along a line; show that their common C.G. is unaltered by the removal of the third particle.

10. On a uniform straight bar 5 ft. long, and whose weight is 5lbs, bodies whose weights are 1, 2, 3, 4lbs. are hung by

strings at distances 1, 2, 3, 4 feet respectively from the extremity; find the distance from the centre of the bar, of the fulcrum, on which the whole will rest.

Ans. 4 in.

11. A right-angled triangular lamina is suspended from its right angle, and the inclination of the hypothenuse to the horizon is 40°; find the acute angles of the triangle.

Ans. 25° and 65°.

12. A triangular lamina ACB is suspended from the angular point C; show that AB is inclined to the horizon at an angle of 30°, if—

$$CA : AB : AC : : 1 : 2 : \sqrt{3}.$$

13. If three equal uniform rods be rigidly united so as to form half of a regular hexagon, prove that if the system be suspended from one of the angles, one of the rods will be horizontal.

14. A triangular lamina, whose sides AB, AC are equal and each double of BC, swings freely about the angular point C; show that the sine of angle which AB makes with the horizon is $\sqrt{\frac{3}{8}}$.

15. From the fact that a system of particles has one centre of gravity only, show that the lines joining the middle points of opposite sides of any plane quadrilateral bisect each other.

16. Prove the following geometrical construction for the C.G. of any quadrilateral area. Let E be the intersection of the diagonals, and F the middle point of the line joining their middle points; draw EF, and produce it to G, making FG equal to one-third of EF; G shall be the C.G. required.

17. From a uniform circular area is cut out another circular area, the latter being described on a radius of the former as diameter; find the C.G. of the remainder.

Ans. The distance of the C.G. from the centre of the larger circle is one-sixth of the radius.

18. Let a and b denote the diameters of two non-concentric spheres forming a shell, and let D denote the distance between their centres; show that the distance of the C.G. of the shell from the centre of the larger sphere is

$$\frac{b^3D}{a^3-b^3}.$$

19. The middle points of two adjacent sides of a square lamina are joined, and the triangle formed by this straight line and the edges is cut off; find the distance of the C.G. of the remainder from the centre of the square, the length of a side being *a*.

Ans. $\frac{a\sqrt{2}}{21}$.

20. Find the C.G. of a plane area formed by an equilateral triangle and a square, the base of the triangle coinciding with one of the sides of the square.

Ans. Let *a* equal a side of the square; the C.G. is on the axis of the figure at a distance from the base of triangle equal to $\frac{3}{26}(4-\sqrt{3})a$.

21. Five equal particles are placed at five of the angular points of a regular hexagon; find the distance of their C.G. from the centre of the hexagon, the length of side being *a*.

Ans. $\frac{a}{5}$.

22. $n-1$ equal particles are placed at $n-1$ of the angular points of a regular polygon of n sides; find the distance of their C.G. from the centre of the circumscribing circle, whose radius is *a*

Ans. $\frac{a}{n-1}$.

23. Seven equal particles are placed at seven of the angular points of a cube whose diagonal is *a*; find the distance of their C.G. from the unoccupied angle.

Ans. $\frac{4a}{7}$.

24. Four heavy particles whose weights are 3, 5, 4 and 2 lbs., are placed at the corners of a square whose side is *a*; find the distance of their C. G. from the centre.

Ans. $\frac{a\sqrt{5}}{14}$.

25. Four particles whose weights are 4, 6, 5, and 3 lbs., are placed at the corners of a square plate, whose side is 26 inches and weight 8 lbs.; find the distance of the common C.G. from the centre of the plate.

Ans. $\sqrt{5}$ in.

26. Particles whose weights are 1, 9, 2, 3, 3 and 6lbs. respectively are placed at the angular points, taken in order, of a regular hexagon whose side is a; find their C.G. and its distance from the centre of the hexagon.

Ans. $\frac{a}{4\sqrt{3}}$.

27. The sides of a trangular area are 3, 4 and 5 and its inscribed circle has been removed; find the distance of the centre of gravity of the remainder from the sides 3 and 4 respectively.

Ans. $\frac{8-\pi}{6-\pi}$ and 1.

28. What is the height from the ground of the centre of gravity of a pyramid formed of four eight-inch shells touching each other?

Ans. $4+2\sqrt{\frac{2}{3}}$ in.

29. A circular table is to be supported by three legs meeting it in its circumference, and two of the angles of the triangle formed by its feet are 30° and 45° respectively; will the table stand or fall?

Ans. Fall.

30. A round table stands on three legs placed on the circumference at equal distances; show that a body, whose weight is not greater than that of the table, may be placed on any point of it without upsetting it.

31. A cubical block of uniform material rests on a perfectly rough inclined plane whose inclination to the horizon is gradually increased, an edge of the cube being parallel to the intersection of the inclined plane and a horizontal plane; show that the cube will overturn when the angle of inclination becomes greater than 45°.

32. A regular polygonal prism of 18 faces is placed on a perfectly rough inclined plane with an edge of the prism parallel to the intersection of the plane with a horizontal plane; determine the greatest inclination of the plane consistent with equilibrium.

Ans. 10°.

33. If the prism (in the last question) has n faces, find the greatest inclination of the plane consistent with equilibrium.

Ans. $\frac{180^\circ}{n}$.

34. A right circular cone whose height is h and the radius of whose base is r, is placed on a perfectly rough inclined plane; if α be the greatest inclination of the plane consistent with equilibrium, show that

$$\tan\alpha = \frac{4r}{h}$$

35. Explain why in ascending a hill we appear to lean forwards; in descending, to lean backwards?

36. Why does a person rising from a chair bend his body forward and his leg backward?

37. What is the use of a rope-dancer's pole?

38. A beam 30 feet long balances itself on a point at one-third of its length from the thicker end; but when a body whose weight is 10lbs. is suspended from the smaller end, the prop must be moved two feet towards it, in order to maintain equilibrium; find the weight of the beam.

Ans. 90lbs.

39. A bar, the mass of each foot of whose length is 7lbs., rests upon a fulcrum distant 3 feet from one extremity, what must be its length that a mass of 7½lbs. suspended from that extremity may just be balanced by a mass of 20lbs. suspended from the other?

Ans. 9 ft.

40. A rod, AB, 11 inches long, has a mass of 7 ounces suspended at A, and one of 8 ounces at B, and is found to be in equilibrium when balancing on a fulcrum 5 inches from B. If the masses be interchanged the fulcrum must be shifted $\frac{15}{17}$th of an inch; find the mass of the rod and the position of its C.G.

Ans. 2oz.; 1 inch from fulcrum.

41. A body m suspended from one end of a lever, whose weight is neglected, is balanced by a mass of 1lb. at the other end of the lever; and when the fulcrum is removed through half the length of the lever, it requires a mass of 10lbs. to balance m; determine the mass m.

Ans. 5lbs. or 2lbs.

42. A heavy equilateral triangle, hung upon a smooth peg by a string the ends of which are attached to two of its angular points, rests with one of its sides vertical; show that the length of the string is double of the altitude of the triangle.

43. A weightless string $ACBD$, tied to the end A of a uniform rod AB, whose weight is W, passes through a fixed ring at C, and through a ring at the end B of the rod, the free end of the string supporting a body D, whose weight is P; show that the resultant of the tensions along BC and AC is vertical, and that

$$AC : BC :: 2P+W : W.$$

44. A heavy triangle ABC is suspended successively from the angles A and B and two positions of any side are found to be at right angles to each other; prove that $5c^2 = a^2 + b^2$.

45. If G be the C.G. of a triangular area ABC, show that—

$$3(AG^2+BG^2+CG^2) = AB^2+BC^2+CA^2$$

46. If G be the C.G. of a triangle ABC and O any point in the plane of the triangle, show that—

$$OA^2+OB^2+OC^2 = AG^2+BG^2+CG^2+3OG^2$$

47. If G be the C.G. of a system of particles in one plane, m the mass of any one particle, ρ its distance from G, and r its distance from any point O in the plane, prove that—

$$\Sigma(mr^2) = \Sigma(m\rho^2) + GO^2\ \Sigma(m).$$

48. In the last example show that the sum of the products of the mass of each particle and the square of its distance from a given point is least when the point conicides with the C.G. of the particles.

49. ABC is a triangle, G its C.G. and O any point in its plane; forces are represented in magnitude and direction by OA, OB, OC; show that their resultant is represented by $3\ OG$.

50. Forces are represented in direction by the straight lines drawn from a point to a system of particles lying in one plane, each force being proportional to the product of the length of the straight line and the mass of the corresponding particle; prove that the resultant force is represented in direction by the line drawn from the point to the C.G. of the particles, and is proportional to the product of the length of the line and the sum of the masses of the particles.

CHAPTER VI.

RESISTING FORCES AND PROBLEMS.

131. A surface which opposes no resistance whatever to the motion of a body along it is called a smooth surface; otherwise the surface is said to be *rough*. Art. 23. The nature of the action between surfaces in contact will be considered in this chapter.

The following geometrical definitions are here stated for convenience:—

The plane touching a curved surface at any point is called the *tangent plane* at the point.

The perpendicular to the tangent plane at the point of contact is called the *normal* to the surface at that point.

The surfaces of two bodies which touch have a common tangent plane and a common normal at the point of contact.

132. Reaction of a Smooth Surface.—When a body having one point in contact with a fixed rigid surface is pressed against the surface by given forces, the surface resists, as far as it can, the tendency of the forces to produce motion in the point of the body in contact with the surface. This force of resistance is called the reaction of the surface, and is by Newton's law equal and oppositely directed to the force with which the point of contact of the body presses against the surface. Art. 17. We shall first consider the case when the surfaces in contact are smooth.

Whatever be the direction of the reaction of a surface at any point it can always be resolved into two forces: one along the normal; the other in the tangent plane perpendicular to the normal.

If the surface be smooth the force of resistance along the tangent plane is zero, from the definition of smoothness. Art. 131. Hence—

A smooth surface can only resist pressure in the direction of the normal at the point where the pressure is applied. Art. 23.

Since the surface is supposed to be rigid (Art. 15), the *magnitude* of the reaction of the surface may be any whatsoever, depending on the normal pressure against it (Art. 17), but the *direction* of the reaction can only be along the normal.

In general, when two surfaces are in contact with each other at a single point, they react on each other with equal forces oppositely directed along the common normal. The magnitude of either of these equal forces is equal to the resultant normal force which presses the point of contact of one of the surfaces against the other.

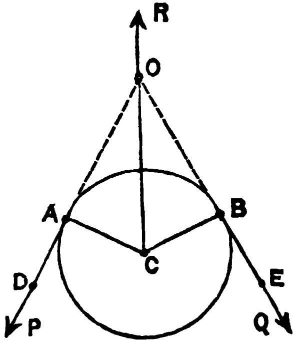
Hence, when a body, acted upon by given forces, is placed with one point in contact with a smooth fixed surface, the body will be in equilibrium when the resultant of the given forces has its line of action coincident with the normal at the point of contact, and presses the body against the surface.

When two bodies have a finite extent of their surfaces in contact, and are in equilibrium, the position and magnitude of the resultant action between them may be determined, when the position and magnitude of the resultant force which presses one against the other is known, from the principle that action and reaction are equal and oppositely directed.

133. *A string is stretched over a smooth circle by forces acting at its extremities, to determine the condition of equilibrium and the resultant reaction of the circle.*

The string is supposed to be flexible and inextensible, and its weight is neglected.

Let $DABE$ be the string in contact from A to B with the circumference of the smooth circle whose centre is C.



Since the radius of a circle is normal to its circumference, the reaction of the surface on the string at every point must pass through the centre C. Hence the resultant reaction R of the circle on the string must pass through the centre C.

Since the string is in equilibrium we may treat it as a rigid body. Art. 16. The forces which maintain equilibrium are the forces P and Q, applied at the ends of the string, whose lines of action are evidently tangents to the circle at A and B, and the resultant reaction of the circle, whose line of action must pass through C and through O, the intersection of the lines of action of P and Q. Art. 63.

By the nature of a circle the angle $AOC = BOC = a$

Hence, considering the equilibrium of the three forces P, Q, R meeting at O, we have, Art. 41,

$$\frac{P}{\sin a} = \frac{Q}{\sin a} = \frac{R}{\sin 2a}.$$

Hence $$P = Q;$$

and $$R = 2P \cos a.$$

The condition of equilibrium is—*The tensions of the portions of the string separated by the circle must be equal.* Art. 22.

The resultant reaction of the circle bisects the angle between the directions of the two portions of the string, which are not in contact with the circle.

We might have shown that the tensions of the portions of the string, separated by the circle, must be equal, when there is equilibrium, without considering the resultant reaction of the circle.

For, taking moments of all the forces acting on the string round the point C, we get, as a necessary condition of equilibrium

$$P.\,AC = Q.\,CB \therefore P = Q.$$

The moment of the reaction of the circle on the string at every point is zero, since the reaction at every point must pass through C.

Obs.—In the proof of this proposition we have treated the forces acting on the flexible string as if they acted on a rigid body. This is lawful (Art. 16), because since the flexible string is in equilibrium, if we supposed it to be replaced by a rigid body of exactly the same form, the same forces would act in the same manner on the rigid string, and would maintain equilibrium. By the principle of Art. 16, we may treat any body which we know to be in equilibrium as a rigid body. The principle is extensively used in Hydrostatics.

134. *When a string is stretched in equilibrium over any smooth surface, the tension of the string is the same throughout.* Art. 22.

The string is supposed to be perfectly flexible and inextensible, and its weight is neglected.

The limits of the present work do not allow us to give a strict proof of this proposition. It may however be regarded as a result of experience, or as an axiom depending on the nature of a string and a smooth surface.

For, if the tensions from point to point along the string were not equal, motion would ensue, because the smooth surface cannot resist the motion of the string along it. Or, considering only the portions of the string separated by the surface, the forces at the ends which stretch the string must be equal, otherwise motion would take place.

An important particular case was proved in the last article, and the magnitude and direction of the resultant reaction of the smooth surface can be determined in the same manner in the general case.

For, the tensions of the two portions of the string separated by the surface being equal, the resultant reaction must bisect the angle between them.

When the portions of the string, separated by the surface, are parallel, the resultant reaction of the surface is parallel, midway between them, and equal in magnitude to twice the tension of the string. This is the theory of the *pulley*, which will be described in the next chapter.

135. CONSTRAINED BODY. — A distinction is usually made between bodies which are *free* to move in any manner and bodies whose motions are restricted.

In considering the equilibrium of a constrained body, it is to be observed that the physical effect of the constraints on the body is the same as the effect of a system of forces, the reactions produced by the constraints, of unknown magnitudes but of known directions when the restraining surfaces are smooth.

Taking into account these resisting forces, along with the moving forces acting on the body, the constrained body is to be treated as if it were a free body, and the equations of equilibrium are to be formed for all the forces.

136. The following axiom is always assumed, although not formally stated, about the action of resisting forces.

Suppose a body is acted on by a number of forces, some moving, some resisting; and suppose that consistently with the conditions that limit the action of the resisting forces, there can be assigned to these forces values which can give equilibrium, then there always will be equilibrium.

More briefly—A system of resisting forces will maintain equilibrium if it can do so. See also Art. 17.

Without this axiom, although we could assert that if there was equilibrium certain conditions would hold, we could not assert, conversely, that if the conditions were satisfied there would be equilibrium.

137. It will be often only necessary to consider cases where the surfaces in contact are of indefinitely small extent; as for instance when a particle, or the end of a rod, is in contact with a plane or sphere. In these cases the contact is between a material particle and a finite surface, and when the surface is smooth, the normal to the finite surface determines the direction of the mutual pressure.

In solving problems the student must be very careful to represent properly the lines of action and directions of the unknown resisting forces, denoting their unknown magnitudes by symbols.

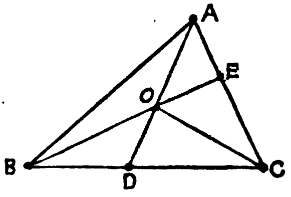
The equations of equilibrium are then to be formed for the entire system of forces, including the unknown reactions. These equations, together with *geometrical equations*, depending on the form of the body and the constraints to which it is subject, will be sufficient for the solution of the proposed problem.

We now give a few examples to illustrate the application of these principles, and the solution of statical problems.

138. Ex. 1.—*A gravitating body rests with three given points in contact with a smooth horizontal plane; to determine the pressures at the three points.*

Let A, B, C be the points at which the body is in contact with the horizontal plane.

Let W be the weight of the body, and let O be the point where its vertical line of action meets the horizontal plane.



The reactions of the plane at A, B, C, which are equal and opposite to the pressures at these points, must be normal to the plane and hence vertical; let their unknown magnitudes be P, Q and R, respectively.

The equilibrium of the body is maintained by the four parallel forces P, Q, R, W; hence the resultant of the like forces P, Q, R must be equal and opposite to W, therefore O is a point on the line of action of the resultant of P, Q and R.

Join AO, and produce it to meet BC in D; then, by the theory of parallel forces, the resultant of R and Q passes through D; hence

$$\frac{Q}{R}=\frac{CD}{BD}=\frac{\text{area } AOC}{\text{area } AOB}.$$

Similarly, it may be shown that

$$\frac{P}{R}=\frac{CE}{AE}=\frac{\text{area } BOC}{\text{area } AOB}.$$

Hence
$$\frac{P}{\text{area } BOC}=\frac{Q}{\text{area } AOC}=\frac{R}{\text{area } AOB}$$
$$=\frac{P+Q+R}{\text{area } BOC+\text{area } AOC+\text{area } AOB}=\frac{W}{\text{area } ABC}.$$

These relations determine the pressures at the points A, B, and C.

This problem might be solved otherwise, analytically, by expressing the condition that the resultant of the three parallel forces P, Q, R, which is equal in magnitude to their sum, should pass through the point O.

Let the co-ordinates of A, B, C, referred to two axes in their plane, be $x,' \; y'$; $x,'' \; y''$; $x,''' \; y'''$; and let the co-ordinates of O be x, y. By Art. 79 we have—

$$Px'+Qx''+Rx'''=Wx;$$
$$Py'+Qy''+Ry'''=Wy;$$
$$P+Q+R=W.$$

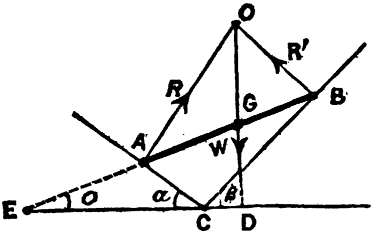
In these three equations the only unknown quantities are P, Q and R, which can hence be determined.

Obs. 1.—This method of solving the problem shows that when the body has more than three points in contact with the plane, the pressures at the points are indeterminate. For, there would be only three equations connecting more than three unknown quantities.

For example, when a table with four legs stands on a horizontal plane, the pressures of the legs on the plane are indeterminate, on the supposition that the table is perfectly rigid. Practically, however, there would be a definite pressure at each leg, because no body in nature is perfectly rigid.

Obs. 2.—In this problem we have supposed gravity to be the only moving force acting on the body. The solution, however, would be the same if W were the resultant moving force acting perpendicularly to the plane through O.

139. Ex. 2.—*A uniform beam rests upon two smooth inclined planes whose line of intersection is horizontal; determine the position of equilibrium of the beam and the pressures on the planes.*



The forces acting on the beam are three in number, the two normal reactions of the inclined planes, and gravity. These forces must lie in one plane for equilibrium. Hence, since the direction of gravity is vertical, the beam must lie in a vertical plane normal to both of the inclined planes and therefore perpendicular to their line of intersection.

Let AC and BC be sections of the inclined planes by the vertical plane in which the beam lies. Let the inclinations of the planes and the beam to the horizon be α, β, θ, respectively.

The known weight W of the beam acts vertically through its middle point, which is its centre of gravity G.

Let the perpendicular reactions of the planes on the beam at A and B, which are equal and opposite to the pressures of the beam on the planes, be R and R', respectively.

The quantities W, α, β being supposed known it is required to determine R, R' and θ; θ fixes the position of equilibrium.

We shall obtain the equations of equilibrium (Art. 99) by resolving parallel to each of the inclined planes, and taking moments round the point G.

Observing that a perpendicular to either of the planes at C would make with the other plane an angle $90° - (\alpha + \beta)$, we have

$$W \sin \alpha = R' \sin (\alpha + \beta); \quad \text{.........}(1)$$

$$W \sin \beta = R \sin (\alpha + \beta); \quad \text{.........}(2)$$

$$R \cos (\alpha + \theta) = R' \cos (\beta - \theta). \quad \text{.........}(3)$$

We have thus three equations between three unknown quantities; hence these equations are the complete solution of the physical problem.

From these equations we obtain

$$\frac{\sin \alpha}{\sin \beta} = \frac{R'}{R} = \frac{\cos (\alpha + \theta)}{\cos (\beta - \theta)}$$

solving for tan θ we get

$$\tan \theta = \frac{\sin (\beta - \alpha)}{2 \sin \alpha \sin \beta}$$

and

$$R = W \frac{\sin \beta}{\sin (\alpha + \beta)};$$

$$R' = W \frac{\sin \alpha}{\sin (\alpha + \beta)}.$$

Obs.—The advantage of resolving parallel to each of the inclined planes is that in each equation thus obtained only one of the unknown quantities appears. It also obviously leads to a simpler equation to take moments round a point situated on the line of action of one of the forces than round any other point. The student should form, as an exercise, the equations of equilibrium by resolving the forces in directions different to those chosen here, and by taking moments round a different point. It will be found that the final results will be always the same, although the elimination may be a little more complicated.

A problem may often be more conveniently solved from geometrical considerations than by forming the equations of equilibrium. The following is a geometrical solution of the same problem:—

Geometrical Solution.—Since the lines of action of three forces which keep a body in equilibrium must meet at a point, the geometrical condition of equilibrium is that the vertical through the centre of gravity G of the beam must pass through the point of intersection O of the lines of action of the reactions of the planes. If this condition be satisfied, the planes will supply the necessary reactions for equilibrium. Art. 136.

From the geometry of the figure it is obvious that

$$\text{angle } AOG = \alpha;\ BOG = \beta;\ BGO = 90 - \theta$$

and
$$\frac{\cos(\alpha + \theta)}{\sin\alpha} = \frac{GO}{GA} = \frac{GO}{GB} = \frac{\cos(\beta - \theta)}{\sin\beta}$$

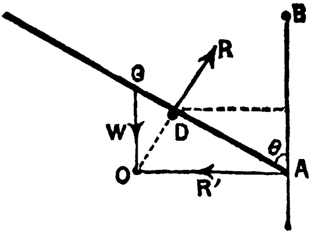
hence
$$\frac{\cos(\alpha + \theta)}{\sin\alpha} = \frac{\cos(\beta - \theta)}{\sin\beta}$$

Thus we obtain, from this geometrical equation, the same value of θ as before.

From the three forces meeting at the point O we can write down immediately, by Art. 41, the values of R and R'.

140. Ex. 3.—*A uniform rod rests over a smooth horizontal rail with its lower end pressing against a smooth vertical wall; determine the position of equilibrium of the rod and the pressures against the wall and rail.*

The forces acting on the rod are three in number, the weight W acting vertically through the centre of gravity G, which is the middle of the rod; the reaction R of the rail D, which is perpendicular to the rod; and the reaction of the wall R', which is perpendicular to the wall.



For equilibrium these forces must lie in the same plane, hence the rod must lie in a vertical plane perpendicular to the wall; the plane of the paper represents this plane.

Let the inclination of the rod to the vertical wall, which determines its position of equilibrium, be θ; let half the length of the rod be a, and the distance of the rail D from the wall b.

To obtain the equations of equilibrium; resolve the forces vertically and parallel to the rod, and take moments round the point A.

$$W = R \sin\theta \text{ ....}(1)$$

$$W\cos\theta = R' \sin\theta \text{ ....}(2)$$

$$W.\ a\sin\theta = R.\ AD = R.\ \frac{b}{\sin\theta} \text{ ......}(3)$$

The quantities a, b and W are supposed known, and we have thus three equations between three unknown quantities, which therefore constitute the complete solution of the physical problem.

From these equations we obtain

$$\sin\theta = \frac{W}{R} = \frac{b}{a\sin^2\theta};$$

therefore $$\sin^3\theta = \frac{b}{a}$$

and $$R = W \operatorname{cosec}\theta$$

$$R' = W\cot\theta$$

GEOMETRICAL SOLUTION.—The lines of action of R, R' and W must meet in a point O, and the geometrical condition for equilibrium is that the vertical through G should pass through the intersection of the perpendiculars to the wall and rod, at A and D, respectively. If this condition be satisfied the wall and rail will supply reactions sufficient for equilibrium. Art. 136.

From the geometry of the figure we have

$$AD = OA\sin\theta;\ b = AD\sin\theta;\ OA = a\sin\theta.$$

Eliminating OA and AD we get, as before,

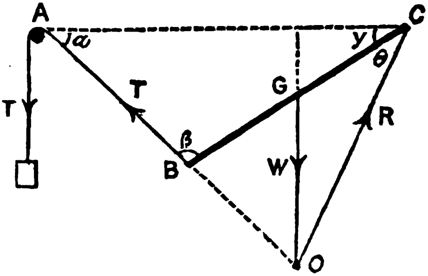
$$\sin^3\theta = \frac{b}{a}$$

The values of R and R' can be immediately written down by Art. 41.

From the geometry of the figure it is evident that equilibrium would be impossible if the distance of the rail D from the wall were greater than half the length of the rod. This also follows from the value for $\sin\theta$, which would be an impossible one if b were greater than a.

141. Ex. 4.—*A uniform rod is capable of turning freely in a vertical plane round a hinge at one extremity: a cord attached to the other extremity passes over a smooth horizontal rail (perpendicular to the vertical plane through the rod, and in the same horizontal plane with the hinge) and supports a body of given mass hanging freely; determine the position of equilibrium of the rod and the direction and magnitude of the pressure on the hinge.*

The rod CB is acted on by three forces; gravity W through its centre of gravity G, which is its middle point; the tension of the cord T, in the direction BA, which is equal to the weight of the given mass and the reaction R, of the hinge at C.



The direction of R is found by joining the point O where the lines of action of T and W intersect to C, since the three forces which maintain equilibrium must meet at a point.

Let the known lengths BC and AC be $2a$ and $2b$, respectively, and the unknown angles of the triangle ABC be α, β, γ. Also let the unknown angle which the direction of the reaction of the hinge OC makes with the horizontal line AC be θ.

To form the equations of equilibrium; resolve horizontally, and perpendicular to AB, and take moments round C.

$$R\cos\theta = T\cos\alpha \text{ ....(1)}$$

$$R\sin(\theta+\alpha) = W\cos\alpha \text{ ....(2)}$$

$$T\,2a\sin\beta = W\,a\cos\gamma \text{ ......(3)}$$

These, which are the only dynamical equations of equilibrium, are not sufficient for the solution of the problem, since these are five unknown quantities R, α, β, γ, θ.

Two additional geometrical equations can however be obtained from the figure; they are

$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta} \text{ ....(4)}$$

$$\alpha+\beta+\gamma = 180^\circ \text{ ....(5)}$$

These five equations are sufficient to determine the unknown quantities, and hence are the complete solution of the physical problem. The determination of the unknown quantities from them is merely a mathematical question.

When these unknown quantities are determined, γ gives the position of equilibrium of the rod, θ the direction of the reaction of the hinge, and R the magnitude of this reaction.

Obs. 1.—If the question asked were to find only the position of equilibrium of the rod, the equations (3), (4), and (5) would be sufficient.

Obs. 2.—The problem might be varied by supposing the string to be fastened to a fixed point at A instead of passing over the rail at A and supporting the given mass. We should have the same equations of equilibrium as before, but α, β, γ, become known quantities, since the triangle ABC would then be a known triangle. The unknown quantities become T, R and θ, and the equations (1), (2), and (3) would be sufficient for their determination.

Obs. 3.—When the rod BC lies above the horizontal line AC, making with it an angle less than a right angle, it is evident that equilibrium could not be maintained by means of a string attached to B and a fixed point at A. The equation of moments (3) would in fact give a negative value for the tension of string, which is impossible, because a string can only resist in the direction which stretches it. In this case (and in similar cases) equilibrium could still be maintained by substituting for the string a rigid rod attached to A and B, as the tension of a rigid rod may be positive or negative. Art. 22.

142. FRICTION.—A rough surface is one which opposes resistance to the motion of a body along it; the tangential resisting force is called the *force of friction.* When a body in contact with a rough fixed surface is in equilibrium, the force of friction at any point of contact is equal and opposite to the resultant tangential force which tends to produce motion at that point. We shall only consider cases (1) when the surfaces of contact are plane, (2) when the contact is at a definite point, (3) when it is along a definite line.

143. When a body rests against a fixed rough surface, the surfaces of contact being plane, the friction developed depends upon the resultant tangential force, which tends to cause the body to slide along the surface, to which it is equal and opposite. If this tangential moving force be gradually increased, a gradually increasing amount of friction is brought into action to resist it, up to a certain maximum and limiting value, depending on the nature of the surfaces in contact and the normal pressure between them. When the moving force is further increased equilibrium is broken.

When the body is in a state *bordering on motion* friction has its *maximum value,* and is then equal and opposite to the tangential force which is *just* sufficient to cause the body to slide.

144. LAWS OF FRICTION.—The following laws of the limiting value of the friction between two solid bodies, when

the surfaces in contact are plane, have been determined experimentally.

(1.) *Friction is directly proportional to the normal pressure between the bodies in contact.*

(2.) *Friction is independent of the extent of the surfaces in contact.*

(3.) Friction depends on the nature of the bodies in contact; upon their polish; on the species and quantity of lubricant which may have been applied.

(4.) Friction increases up to a certain maximum, with the duration of the contact.

These are the laws of statical friction when the body is in a state bordering on motion; that is, when friction has assumed its maximum value. When the statical friction has been overcome, and sliding takes place, friction still acts as a resisting force, but the Kinetic friction is less than the Static friction, and is approximately the same whatever be the rate of motion.

Obs.—These laws are not accurately true in certain exceptional cases. The discussion of these cases is however not suited to the limits of the present work, and we shall assume the laws to be generally true. It will be only necessary to consider laws (1) and (2).

145. COEFFICIENT OF FRICTION.—By law (1) the ratio of the friction to the normal pressure, when the body is in a state bordering on motion, is constant. This constant ratio is called the *coefficient of friction.*

The numerical value of the coefficient of friction depends on the circumstances of the case, and must be obtained from some experimental determination. In practical cases it varies from about 0.03 to 0.80, according to the nature of the surfaces in contact.

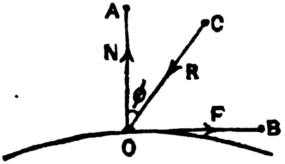
The friction F between two bodies may have any magnitude not greater than μN, where N is the whole normal pressure between the bodies, and μ the coefficient of friction. The direction of the friction F is always such as to oppose directly the tendency of one body to slide along the other.

When the bodies are in a state bordering on motion $F = \mu N$

Obs.—Theoretically we might suppose a surface to be so rough as to prevent all sliding along it. Such a surface is said to be *perfectly rough*, and in this case $\mu = \infty$. A smooth surface would be denoted by $\mu = 0$.

146. Reaction of a rough surface.—The reaction of a rough surface is the resultant of the friction and the normal reaction.

Let O be a particle at rest on a rough surface, and let R be the resultant moving force on the particle in the direction CO; let angle $COA = \phi$.



R, which is equal and opposite to the reaction of the surface, may be resolved into two forces, one along the normal, which is counteracted by the normal reaction N of the surface; the other along the tangent to the surface, in the plane AOC, which tends to make the particle slide along the surface and is resisted by the friction F.

Hence
$$N = R\cos\phi \quad\ldots\ldots\ldots(1)$$
$$F = R\sin\phi \quad\ldots\ldots\ldots(2)$$

From these equations we get

$$F = N\tan\phi$$

The greatest value that F can have is μN; hence $\tan\phi$ cannot be greater than μ.

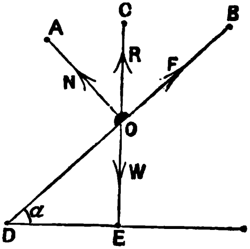
So long as $\tan\phi$ is not greater than μ, the surface will supply a resistance of any magnitude in the direction OC. Art. 136. Hence.—

The reaction of a rough surface at any point may be of any magnitude, but its direction must not lie outside the surface of a cone whose vertex is the point, whose axis is the normal, and whose semi-vertical angle is the angle whose tangent is equal to the coefficient of friction.

Def. 1.—The angle whose tangent is equal to the coefficient of friction is called the *angle of friction*. If $\phi =$ the angle of friction, $\phi = \tan^{-1}\mu$.

Def. 2.—The cone, defined above, is called the *cone of resistance.*

147. *To determine the greatest inclination of a rough inclined plane upon which a gravitating particle will rest without sliding down.*



The figure represents the forces acting on the particle O; the reaction R of the rough inclined plane, which is the resultant of the friction and normal reaction, counteracts the weight W of the particle.

Hence R must be vertical, and the angle COA, which its direction makes with the normal OA, cannot be greater than the angle of friction ϕ.

From the figure it is evident that angle $COA =$ angle $ODE =$ inclination of plane to horizon $= \alpha$.

Hence for equilibrium the inclination α of the plane to the horizon cannot be greater than $\phi = \tan^{-1}\mu$.

The question may be also solved without compounding F and N into their resultant R.

Resolve the forces parallel and perpendicular to the plane.

$$F = W \sin \alpha\,; \quad N = W \cos \alpha.$$

The greater the inclination α, the greater will F be, and its maximum value is $F = \mu N$. From these equations we get

$$\tan \alpha = \mu \quad \therefore \alpha = \tan^{-1}\mu = \phi.$$

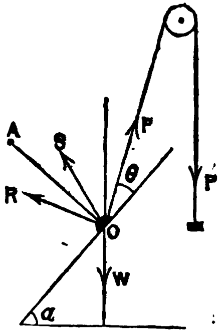
Obs.—The angle of friction, which is the greatest inclination of an inclined plane upon which a gravitating particle can rest, is sometimes called the *angle of repose.*

148. *To determine the coefficient of friction between two substances experimentally.*

Suppose DB (Fig. Art. 147) to be an inclined plane capable of turning round D. Let one of the substances be formed into a plate, and fastened to the inclined plane parallel to it. Let a plane surface of the other substance be placed in contact with this plate.

If now the inclination of the plane DB to the horizon be gradually increased, the tangent of the angle of inclination when the body just begins to slip will be the coefficient of friction, which can thus be determined.

149. *A body of weight W is supported on a rough plane, of given inclination α, by a force acting in a given direction; to determine the limits of the magnitude of the force for which equilibrium is possible.*



The force is supposed to act in a vertical plane perpendicular to the inclined plane. In the figure it is represented as applied by means of a string passing over a smooth pulley and sustaining a body, whose weight is P, hanging freely. The direction of the string, that is, of the force P, makes an angle θ with the plane.

Let the greatest value which P can have, consistent with equilibrium, be P_1. The body is then in a state bordering on motion *up* the plane, therefore the friction acts *down* the plane, and the resultant reaction R of the plane makes with the normal OA an angle equal to the angle of friction, $\phi = \tan^{-1}\mu$.

The forces P_1, W and R maintain equilibrium; hence, by Art. 41,

$$\frac{P_1}{\sin(RW)} = \frac{W}{\sin(RP_1)} \therefore \frac{P_1}{\sin(\alpha+\phi)} = \frac{W}{\cos(\theta-\phi)};$$

or

$$P_1 = W\frac{\sin(\alpha+\phi)}{\cos(\theta-\phi)}.$$

Secondly: Let the least value of P, which is consistent with equilibrium, be P_2. The body is then in a state bordering on motion down the plane; therefore friction acts up the plane, and the resultant reaction S of the plane makes with the normal OA an angle equal to the angle of friction.

The forces P_2, W and S maintain equilibrium; hence, by Art. 41,

$$\frac{P_2}{\sin(SW)} = \frac{W}{\sin(SP_2)} \therefore \frac{P_2}{\sin(\alpha-\phi)} = \frac{W}{\cos(\theta+\phi)};$$

or

$$P_2 = W\,\frac{\sin(\alpha-\phi)}{\cos(\theta+\phi)}.$$

For equilibrium P must not be less than P_2 and not greater than P_1.

From the value of P_2 given above it appears that $P_2 = 0$, when $\alpha = \phi$. For this value of the inclination, and smaller values, the body would be supported by friction alone. See Art. 147.

Obs. 1.—The limiting values of P may be written in one formula,

$$P = W\,\frac{\sin(\alpha\pm\phi)}{\cos(\theta\mp\phi)}$$

and either can be got from the other by changing the sign of ϕ.

Obs. 2.—The student should work this question by resolving forces, as in Art. 147, without compounding the friction and normal reaction, the results will be

$$P = W\,\frac{\sin\alpha\pm\mu\cos\alpha}{\cos\theta\pm\mu\sin\theta}$$

These results become the same as those given above if we write $\tan\phi$ for μ.

150. *To determine the direction of the least force which would bring a body into a state bordering on motion up an inclined plane; that is, to determine the best angle of traction.*

P_1 being the value of the force when the body is in a state bordering on motion up the plane, we have, by the last article,

$$P_1 = W\,\frac{\sin(\alpha+\phi)}{\cos(\theta-\phi)}$$

In this value θ is the only variable quantity, and hence P_1 is least when $\cos(\theta-\phi)$ is greatest, that is, when $\theta=\phi$.

Hence, the best angle of traction is the angle of friction, as the smallest increase to P_1 would produce motion up the plane.

151. *To determine the direction and magnitude of the least force which will support a body on an inclined plane whose inclination is greater than the angle of friction.*

The least force, at an angle θ to the plane, which will support the body is (Art. 149)—

$$P_2 = W\frac{\sin(\alpha-\phi)}{\cos(\theta+\phi)}.$$

In this value θ is the only variable quantity, and hence P_2 is least when $\cos(\theta+\phi)$ is greatest, that is, when $\theta=-\phi$.

The direction of the force must therefore make with the plane, below the plane, an angle equal to the angle of friction.

The magnitude of least force is $W\sin(\alpha-\phi)$.

EXAMPLES.

1. A particle, whose weight is 2 W, is sustained upon a smooth inclined plane by three forces each equal to W, one acting vertically upwards, another along the plane, and the third parallel to the horizon; show that the inclination of the plane to the horizon is $2\tan^{-1}\frac{1}{3}$.

2. Two particles, whose weights are P and Q, support each other on two smooth inclined planes, making angles with the horizon α and β respectively, by means of a string passing over the common vertex of the inclined planes; find the ratio of P to Q and the tension of the string.

Ans. $T = P\sin\alpha = Q\sin\beta$.

3. Two particles, whose weights are P and Q, are connected together by a fine thread passing over a smooth pulley; the particle whose weight is P rests on a smooth inclined plane, inclination α, and the other particle hangs freely; determine

the position of equilibrium and the pressure on the inclined plane.

Ans. Let θ be the inclination of the string to the plane, R the pressure on plane—

$$Q \cos \theta = P \sin \alpha ;$$

$$R = P \cos \alpha - (Q^2 - P^2 \sin^2 \alpha)^{\frac{1}{2}}$$

4. A particle, whose weight is $2P$, is supported on a smooth inclined plane by two equal forces P, P, one acting along the plane, and the other in a direction inclined to the plane at an angle equal to the inclination of the plane; show that the inclination of the plane is $\sin^{-1}\frac{4}{5}$.

5. A sphere, whose radius is r and weight W, rests upon two upright posts of equal heights and at a distance apart equal to $2a$; find the horizontal thrust upon each of the posts.

Ans. $\dfrac{Wa}{2\sqrt{r^2 - a^2}}$

6. A roof ACB is wholly composed of beams forming isosceles triangles, of which AB is the base; find the horizontal thrust on each of the side walls, the weight of the roof being W and θ the angle which it makes with the horizon.

Ans. $\frac{1}{2} W \cot \theta$.

7. Two rafters, AB and AC, are each 25 feet long, and their feet are tied by an iron rod BC, whose length is 40 feet; a body of weight, 1 ton, is suspended from A; determine the horizontal strain which it produces on the tie neglecting the weight of the rafters.

Ans. 2986⅔ lbs.

8. A sphere, whose weight is W, rests on two smooth inclined planes, whose inclinations are α and β; find the pressure on each plane.

Ans. $\dfrac{W \sin \beta}{\sin(\alpha + \beta)}$; $\dfrac{W \sin \alpha}{\sin(\alpha + \beta)}$.

9. A carriage wheel, whose weight is W and radius r, rests upon a level road, show that the horizontal force P, applied at the centre, which is necessary to draw the wheel over an obstacle of height h is—

$$P = W \frac{\sqrt{2rh - h^2}}{r - h}.$$

10. A string fastened at two points, not in the same horizontal line, supports a body, whose weight is W, by means of a ring which slides on the string; if a be the distance between the points, ϕ the angle which the line joining them makes with the vertical, l the length of the string, show that the tension of the string is determined by the following equations:

$$2T\cos\alpha = W;$$

$$l\sin\alpha = a\sin\phi.$$

11. If the ring in the last example be fastened to a fixed point in the string, show how to find the tensions of the two portions of the string.

12. A beam is suspended by two strings attached to fixed points in the beam, and to a point above the beam, show how to find the inclination of the beam to the horizon in the position of equilibrium, and the tensions of the strings.

13. A beam is suspended by a string attached to fixed points in the beam, the string passing over a smooth peg; show how to find the inclination of the beam to the horizon in the position of equilibrium and the tension of the string.

14. A smooth circular ring is fixed in a horizontal position, and a small ring, P, sliding upon it is in equilibrium when drawn by two strings in the directions PA, PB; show that if PC be a diameter of the circle the tensions of the strings are in the ratio of BC to AC.

15. Two forces, P and Q, acting in the diagonals of a parallelogram, whose weight is W, keep it at rest in such a position that one of its edges is horizontal; if α and β be the angles between its diagonals and the horizontal side, show that

$$P\sec\alpha = Q\sec\beta = W\operatorname{cosec}(\alpha+\beta).$$

16. A beam, whose length is $2l$, and whose centre of gravity is at a distance a from one end, rests entirely within a smooth hemispherical bowl of radius r; find the inclination of the beam to the horizon.

Ans. $\tan^{-1}\dfrac{l-a}{\sqrt{r^2-l^2}}$.

17. A uniform rod, whose length is $2a$, rests partly within and partly without a smooth hemispherical bowl of radius

r: if θ be the inclination of the rod to the horizon in the position of equilibrium, prove that

$$4r^2 \cos^2 \theta - a \cos \theta - 2r = 0$$

18. Show that no uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl at an inclination to the horizon greater than $\sin^{-1}\sqrt{\tfrac{1}{3}}$.

19. A uniform circular lamina is placed with its centre O upon a prop; particles, whose weights are P, Q, R, are placed at points A, B, C on its circumference, so that the lamina remains horizontal; show that

$$\cos AOB = \frac{R^2 - P^2 - Q^2}{2PQ};$$

$$\cos BOC = \frac{P^2 - Q^2 - R^2}{2QR};$$

$$\cos COA = \frac{Q^2 - P^2 - R^2}{2PR}.$$

20. The lid of a desk, whose width is 15 inches and weight 6lbs., is held in a horizontal position by a cord, 30 inches long, fastened to the edge of the desk, and to a point vertically over the hinge; required the tension of the cord.

Ans. $2\sqrt{3}$ lbs.

21. A uniform door, whose width is $2a$ and weight W, is suspended from two hinges in a vertical line at a distance apart $2b$, and situated symmetrically with respect to the C.G. of the door; show how to find the directions of the strains on the hinges and that their magnitudes are given by

$$R = R' = \frac{W}{2b}\sqrt{a^2 + b^2}.$$

22. A beam is moveable in a vertical plane about a hinge at one end, and the other end is supported by a string fastened to a fixed point in the vertical plane through the beam; show how to find the direction and magnitude of the strain on the hinge and the tension of the string.

23. A lever (whose weight is neglected) in the form of an arc of a circle subtending an angle 2α at the centre, and having two bodies, whose weights are P and Q, suspended from its extremities, rests with its convexity downwards upon a horizontal plan

if θ be the inclination of the line joining the ends of the lever to the horizon, prove that

$$\tan\theta = \frac{P-Q}{P+Q}\tan\alpha.$$

24. Two small rings, whose weights may be neglected, slide on the arc of a smooth vertical circle; a string passes through both rings and has three equal heavy particles attached to it, one at each end, and one between the rings; show that in the position of equilibrium the distance between the rings is equal to the radius of the circle.

25. An inextensible string binds tightly together two smooth cylinders of radii r and r'; find the ratio of the mutual pressure R between the cylinders to the tension T by which it is produced.

Ans. $\frac{R}{T} = \frac{4(rr')^{\frac{1}{2}}}{r+r'}$.

26. Two equal balls are placed within a hollow vertical cylinder, open at both ends, which rests upon a horizontal plane; the weight of each ball is W and radius r, the radius of the cylinder being r'; find the least value W' of the weight of the cylinder in order that it may not be upset by the balls.

Ans. $W' = 2W\left(1 - \frac{r}{r'}\right)$.

27. One end of a string is fixed to the extremity of a smooth uniform rod, and the other to a ring without weight, which passes over the rod, and the string is hung over a smooth rail; prove that if α be the inclination of the rod to the horizon in the position of equilibrium, $2l$ the length of the string and $2a$ the length of the rod.

$$l(\tan^3\alpha + \tan\alpha) = a.$$

28. In the last problem show that the inclination of the rod to the vertical cannot be less than 45°, and find the least length of the string for which equilibrium is possible.

Ans. $2l = a$.

29. *The funicular polygon.*—The ends of a string are fastened to two fixed points, and from knots in the string bodies of given weight are suspended; show that the horizontal component of the tension is the same for all portions of the string. Prove also that, when the weights of the bodies are all equal, the tangents of the angles which the successive portions of the string make with the horizon are in Arithmetical Progression.

FRICTION.

1. A body, whose weight is 10lbs., rests on a rough plane inclined to the horizon at an angle of 30°; find the normal pressure, *R*, on the plane and the force, *F*, of friction exerted.

Ans. $F = 5$lbs.
$R = 5\sqrt{3}$ lbs.

2. If the roughness of a plane, which is inclined to the horizon at an angle α be such that a body, whose weight is W, will just rest supported on it; find the least force along the plane required to drag the body up.

Ans. $2W \sin \alpha$.

3. A pair of carriage wheels, which can turn without friction round a common axle, are placed on a rough inclined plane with the axle parallel to the line of intersection of the plane with the horizon; determine the force, applied at the centre of the axle parallel to the plane, which will support the system, the weight of the system being W and α the inclination of the plane.

Ans. $W \sin \alpha$.

4. P is the lowest point on the convex circumference of a rough cylinder at which a particle can rest, friction being equal to the pressure; find the inclination of the radius through P to the horizon.

Ans. 45°.

5. A ladder, 10 feet long, rests with one end against a smooth vertical wall and the other on the ground, the coefficient of friction being $\frac{1}{2}$; find to what height from the ground a man, whose weight is four times that of the ladder, may ascend before it begins to slip, the foot of the ladder being 6 feet from the wall.

Ans. $5\frac{5}{8}$ feet.

6. A heavy body, whose weight is W, is to be conveyed to the top of a rough inclined plane, whose inclination is α and coefficient of friction μ; show that it will be easier to lift the body vertically or to drag it along the plane according as

$$\mu < \text{or} > \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$$

7. A rough plane is inclined to the horizon at an angle of 60°; find the magnitude and direction of the least force which

will prevent a body weighing 100 lbs. from sliding down the plane, the coefficient of friction being $\sqrt{\frac{1}{3}}$

Ans. 50lbs.

8. A square board $ABCD$, the plane of which is vertical, rests with its side AD in contact with a rough vertical wall, which is perpendicular to the plane of the board; the side AB rests at a point indefinitely near to B, upon a rough peg; find the least value of the angle of friction, supposing it to be the same for the wall and for the peg.

Ans. 27° 30′.

9. A beam rests between the ground and a vertical wall; find the angle θ which the beam makes with ground when in a state bordering on motion.

Let a, b be the segments of the beam made by its C. G.; μ and μ' the coefficients of friction with the ground and the wall.

Ans. $\tan\theta = \dfrac{a - \mu\mu' b}{\mu\,(a + b)}$.

10. A uniform and straight plank, length $2a$, rests with its middle point on a rough horizontal cylinder, radius r, their directions being perpendicular to each other; find the weight P of the greatest mass which can be suspended from the other end of the plank without its sliding off the cylinder, the weight of the plank being W, and the coefficient of friction $\tan\phi$.

Ans. $\dfrac{P}{W} = \dfrac{r\,\phi}{a - r\,\phi}$.

11. A uniform rod rests within a rough circle the plane of which is vertical; if $180^\circ - 2a$ be the angle subtended by the rod at the centre of the circle, ϕ the angle of friction, and θ be inclination of the rod to the horizon when the friction can only just maintain equilibrium, show that—

$$2\tan\theta = \cot(a - \phi) - \cot(a + \phi).$$

CHAPTER VII.

MACHINES.

152. **Machines.**—A machine may be defined as an instrument, or a system of bodies, by means of which force may be transmitted from one point to another, and altered both in magnitude and direction.

There are certain *Simple Machines* or, as they are called, *Mechanical Powers*, by the combinations of which all machines, however complex, are constructed. These are usually considered to be six in number; the Lever, the Wheel and Axle, the Pulley, the Inclined Plane, the Wedge, and the Screw.

In discussing these machines we shall suppose them to be at rest. When two forces acting on a machine *balance* each other, one of them is called, for convenience, the *power*, we shall denote it by P; the other is the *external resistance* (or the *weight*), we shall denote it in general by W.

The general problem to be considered is to investigate the magnitude of the power which balances, by means of the system of bodies which compose the machine, a given external resistance. The fixtures of the machine counteract the resultant of the power and resistance. To simplify the investigation we suppose the bodies which compose the machine to be perfectly smooth, and their weights will be neglected except when the contrary is stated.

Def.—The ratio $W : P$ is sometimes called the *Modulus* of the machine. When the modulus is greater than unity, the machine is said to work at a *mechanical advantage*, otherwise not.

Obs.—The meaning of this advantage and the functions of a machine will be more fully considered in the next chapter.

153. **The Lever.**—A *lever* is a rigid body, or a rigid rod, which is capable of turning freely in one plane round a

fixed point, or fixed axis, called the fulcrum. The parts into which the fulcrum divides the lever are called the *arms*. When the arms are in the same straight line, the lever is called a straight lever, otherwise a bent lever.

154. *To determine the condition of equilibrium in the lever and the pressure on the fulcrum.*

Let the power applied at the extremities of one arm be P, and the resistance at the extremity of the other arm W. These forces are supposed to lie in the same plane perpendicular to the axis of the lever. Let the position of the fulcrum be denoted by F.

First, let the weight of the lever itself be neglected. The lever is kept in equilibrium by three forces P, W, and the reaction R of the fulcrum. The resultant of P and W, which is the pressure on the fulcrum, must be equal and opposite to R, and its line of action must therefore pass through the fulcrum. Hence the condition of equilibrium is—*the moments of P and W round the fulcrum must be equal and opposite.* See also Art. 87.

The pressure on the fulcrum is equal and opposite to R, and its direction may be found by joining the intersection of P and W, when they are not parallel, to F. Let θ be the angle between P and W, then

$$R^2 = P^2 + W^2 + 2PW\cos\theta$$

When P and W are parallel, R is a parallel force passing through the fulcrum and is equal in magnitude to their sum or difference.

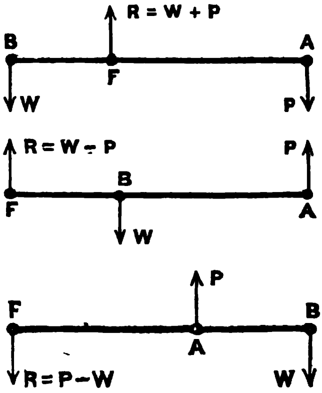
When the weight w of the lever is taken into account, the condition of equilibrium is—*the algebraic sum of the moments of P, W and w round the fulcrum must be equal to zero.* The pressure on the fulcrum may be determined in magnitude and direction by the method of Art. 99.

155. THREE KINDS OF LEVERS.—Levers are divided into three classes according to the relative positions of the points at which the lines of action of P and W intersect the lever, and the fulcrum F.

In the first kind F is intermediate.

In the second kind W is intermediate.

In the third kind P is intermediate.



The diagrams exhibit straight levers of the three kinds, when the forces are parallel, and show the direction and magnitude of the reaction of the fulcrum, which is equal and opposite to the pressure upon it, in each case.

The arms of the lever are in each case FA and FB, and the condition of equilibrium is

$$P.FA = W.FB.$$

This condition may be expressed in words.—*The power is to the resistance inversely as the arms of the lever.*

Considering the three forces P, W, R, which keep the lever at rest, the condition of equilibrium may be expressed by taking moments round *any* point in the plane of the lever. One of the points A, F, B will generally be the most convenient.

It is evident that in a lever of the first kind there may or may not be mechanical advantage; in the second kind there is always mechanical advantage; and in the third kind there cannot be mechanical advantage.

Obs.—The relative positions of F, W and P in the three kinds of levers may be easily remembered by the following "jingle."

F, W, P,
Of levers three,
Must each in turn
The centre be.

156. **Examples of Levers.**—A crowbar employed to move a heavy stone is a lever of the first kind, the fulcrum used being generally a smaller stone. Scissors, nippers and pincers are double levers of this kind.

The oar of a boat is an example of a lever of the second kind, the water acts as a fulcrum; the power is applied at the handle of the oar, and the resistance to be overcome is at the rowlock of the boat. A pair of nutcrackers is a double lever of this kind.

The human forearm is an example of a lever of the third class; the fulcrum is at the elbow; the power is exerted by a muscle connecting the upper part of the arm with the forearm near the elbow; the resistance is the weight of the object raised in the hand. A pair of tongs used to hold a coal of fire is a double lever of this class.

157. **The Common Balance.**—The common balance is a lever of the first kind with equal arms. Its use is to determine the quantity of matter or mass of a body, the masses of two bodies being assumed to be equal when they gravitate towards the Earth with equal forces at the same place, and the process is called weighing.

It consists of a beam with a scale-pan suspended from each extremity; the beam can turn round a fulcrum which is vertically above the common centre of gravity of the beam and the scales when the beam is horizontal. Hence, when the scales are empty, the position of equilibrium of the beam is horizontal.

The substance whose mass is to be determined is placed in one scale-pan. Known standards of mass are placed in the other scale-pan, sufficient in amount to make the beam rest in a horizontal position. In this position the vertical through the centre of gravity of the beam and scales passes through the fulcrum. Let W be the gravitating force of the substance and P that of the standards in the other scale. The moment of W round the fulcrum is equal to the moment of P; hence, since the arms of the lever are equal, $W=P$. Therefore the mass of the substance is equal to the known mass of the standards.

Obs. 1.—The British standard of mass is the mass of an imperial pound avoirdupois, and a balance with a set of standards of mass, which are multiples or sub-multiples of this unit, will serve to determine the mass of a body in any locality whatsoever. If we wish to determine further the force with which the body gravitates in a given locality (or, the weight of the body) we must determine the force with which the unit of mass gravitates at the place. The method of making this determination will be explained hereafter. It has been already stated (see Art. 10) that if g be the force with which one pound of matter gravitates at the place, the gravitating force W of a body whose mass is m pounds is

$$W = mg.$$

Obs. 2.—If a balance be a *true* one, the beam should *remain* horizontal when the scales are empty. The centre of gravity of the beam and scales must then be vertically *below* the fulcrum. The arms of the beam must also be of equal length. If these two conditions be satisfied, and not otherwise, the beam will remain horizontal when equal masses are placed in each scale-pan. When the above conditions are not satisfied, the balance is said to be *false.* A false balance may be detected by interchanging the contents of the scale-pans: for, if the balance be false, the beam cannot remain horizontal when the substances placed in the pans are interchanged.

158. The requisites for a good balance are the following:

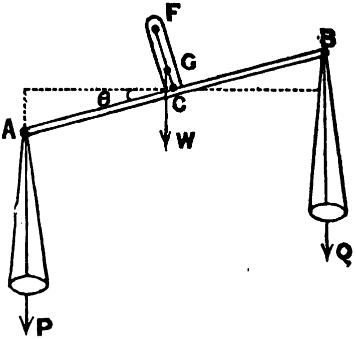
(1.) When equal masses are placed in the scales the beam should be horizontal; that is, the balance should be *true.*

(2.) When the contents of the scale-pans differ in mass, by even a small quantity, the beam should make a considerable angle with the horizon, so that the difference of mass may be easily detected: that is, the balance should be *sensible.*

(3.) When the balance is disturbed, it should tend to return to its position of equilibrium with considerable force: that is, the balance should be *stable.*

159. *To show how the requisites of a good balance may be satisfied.*

It is assumed that the line joining the points of attachment of the scale-pans to the arms is at right angles to the line joining the centre of gravity of the beam with the fulcrum.



Let the weights of the mass of the beam and the masses in the pans be W, P, Q respectively. Let G be the centre of gravity of the beam; F the fulcrum; $AC = BC = a$; $FG = k$; $FC = h$ = distance of F from AB.

Suppose P to be greater than Q, and the inclination of beam to horizon θ

Taking moments about the fulcrum F we have for equilibrium the equation

$$P(a\cos\theta - h\sin\theta) = Q(a\cos\theta + h\sin\theta) + Wk\sin\theta$$

$$\therefore \tan\theta = \frac{(P-Q)a}{(P+Q)h + Wk}.$$

The first requisite is satisfied, because when $P = Q$, $\theta = 0$; therefore the beam will be horizontal when equal masses are placed in the pans.

For a given difference of P and Q, the angle θ should be as large as possible, to satisfy the second requisite; hence $(P+Q)\frac{h}{a} + W\frac{k}{a}$ must be as small as possible, therefore $\frac{h}{a}$ and $\frac{k}{a}$ must be small.

When P and Q are equal, and when the beam is displaced from its position of equilibrium, the moment of the couple which tends to restore the equilibrium is

$$Q(a\cos\theta + h\sin\theta) + Wk\sin\theta - P(a\cos\theta - h\sin\theta)$$

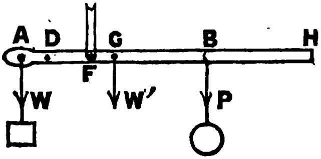
or

$$(2Ph + Wk)\sin\theta.$$

Hence, to satisfy the third requisite, this moment should be as large as possible for a given value of θ, therefore h and k should be large.

Thus the conditions of sensibility and stability are, in part, at variance with each other. They may both, however, be satisfied by making h and k considerable, but a much larger than either, so that the ratios $\frac{h}{a}$ and $\frac{k}{a}$ shall still be small; that is, the distances of the fulcrum from the beam and from the centre of gravity of the beam should be considerable, but the length of the arms much greater than either distance.

160. THE COMMON STEELYARD.—The common steelyard is a balance in which the arms of the lever are unequal, and the same mass is used to weigh different bodies by changing its distance from the fulcrum.



AH is a heavy beam moveable about a fulcrum F. From a point A on the shorter arm the substance to be weighed is suspended. In practice a hook, or a scale-pan, attached at A, serves to hold the substance. A ring B carrying a mass of weight P can be made to slide along the arm FH until the lever is horizontal, the forces then balancing round the fulcrum.

Let W be the weight of the substance, W' the weight of the beam and scale-pan acting at their common centre of gravity G. For equilibrium, taking moments about F,

$$W.AF = W'.FG + P.FB.$$

This formula may be put into a more convenient shape for graduating the arm of the steelyard.

Take a point D between A and F, such that $P \cdot DF = W'FG$. Thus D is the point where the weight P should act so as to balance the weight of the beam and the scale-pan when the scale-pan is empty. Substituting, we get

$$W \cdot AF = P\,(DF + FB) = P \cdot DB$$

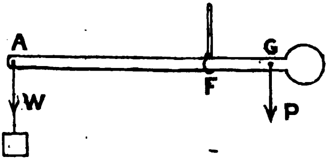
$$W = \frac{DB}{AF} \cdot P$$

The steelyard may be graduated from this formula by taking distances from D successively equal to AF, $2AF$, $3AF$ ...., and these distances may be subdivided so as to suit circumstances.

When the ring is placed at any of these divisions or subdivisions, so that the weight of the known mass suspended from the ring balances the weight of the substance in the scale-pan, the mass of the substance, in terms of the given mass, is determined by reading off the number of divisions and subdivisions on the arm.

Obs.—This balance is sometimes called the *Roman* steelyard. Rumân is an Arabic word for a pomegranate, and the shape of the counterpoise seems to have given rise to the name.

161. *To describe the Danish Steelyard and to show how it is graduated.*



In this balance the fulcrum is a moveable ring which slides along the arm. The beam terminates in a heavy bulb which, by means of the moveable fulcrum, counterpoises the weight of the substance whose mass is to be determined.

Suppose the beam to be horizontal when a substance whose weight is W is suspended by means of a scale-pan, or hook, from A.

Let P be the weight of the beam and scale-pan and G the common centre of gravity. Taking moments about F we have

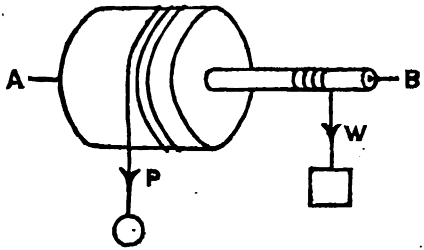
$$W\cdot AF = P\cdot FG = P\,(AG - AF).$$

therefore $$\frac{AF}{AG} = \frac{P}{P+W}.$$

By taking W successively equal to P, $2P$, $3P$ .... we can mark on the beam the corresponding positions of the fulcrum. The subdivisions must be similarly marked by giving to W intermediate values. The mass of any body suspended at A can then be determined in terms of the known mass, whose weight is P, by reading off, on the arm from A, the number of divisions and subdivisions to the point at which the fulcrum is placed when the beam is horizontal.

Cor.—If the successive values of W form an arithmetical progression, the distances of the corresponding graduations from A will form an harmonical progression. This easily follows from the formula for graduation.

162. THE WHEEL AND AXLE.—This machine is a modification of the lever. It consists of two cylinders, rigidly united, having a common horizontal axis, AB, terminating in pivots which rest in fixed sockets (not shown in the diagram) at A and B. The larger cylinder is called the *wheel* and the smaller the *axle*.



The resistance W, which is supposed to be the weight of a body to be lifted, acts by a string which is fastened to the axle and coiled round it. The power P acts by a string which is fastened to the wheel and coiled round it in a direction opposite to that in which the other string is coiled round the axle. Thus the forces tend to turn the machine in opposite directions round the axis AB.

All the forces which act on the machine, except P and W, pass through the axis. Hence we have for equilibrium, taking moments of all the forces round the line AB, Art. 93,

$$P \times \text{radius of wheel} = W \times \text{radius of axle};$$

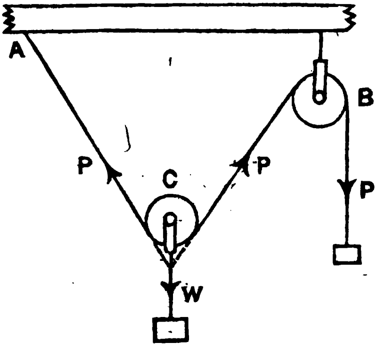
therefore
$$\frac{W}{P} = \frac{\text{rad. of wheel}}{\text{rad. of axle}}.$$

163. THE PULLEY.—A pulley consists of a small circular plate with a groove cut round its edge so as to prevent a string from slipping off when put round the wheel. The wheel is capable of turning freely round an axis passing through the centre of the plate, the extremities of the axis being supported by a frame-work which is called the *block*. The principle upon which the theory of pulleys depends has

been demonstrated in Art. 133. It is there shown that *the tensions of the portions of the same string, which are separated by a smooth circular groove, are equal.*

If the groove were perfectly smooth it would not be necessary for the circular plate to turn freely round its axis; practically it is impossible to have the groove smooth, but it is found that when the plate can turn freely round its axis the tensions of the string are almost exactly the same on both sides of the pulley. We shall always suppose this to be the case. The weights of the strings will be neglected, and, except the contrary be stated, the weights of the pulleys.

164. *To find the ratio between the power and resistance in the single moveable pulley.*



In the figure C represents the moveable pulley. The string is attached to a fixed point at A and after passing under the groove of the moveable pulley C passes again over the groove of a fixed pulley B. To the end of the string is attached a mass whose weight, P, is the power. The resistance is the weight, W, of a body which is attached to the block of the moveable pulley.

The pulley is kept in equilibrium by the tensions of the portions of the string, which are each equal to P, and by the force W acting vertically downwards. Hence the line of action of W must pass through the intersection of the lines of action of the equal forces P, and bisect the angle, $2a$, between these lines. Therefore (Art. 41)

$$\frac{W}{P} = \frac{\sin 2a}{\sin a} = 2 \cos a$$

which is the ratio required.

Otherwise.—Let the angles which the portions of the string make with the vertical be α and α'. Resolving forces vertically and horizontally we have

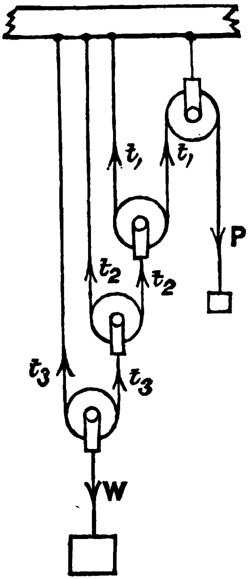
$$P \cos \alpha + P \cos \alpha' - W = 0 \text{ .... } (1)$$

$$P \sin \alpha - P \sin \alpha' = 0 \text{ .... } (2)$$

By (2) we have $\alpha = \alpha'$; substituting in (1) we get the same relation as before.

Cor.—When the portions of the string are parallel we have $\alpha = 0$, hence $W = 2P$. This is also evident, since W is equal and opposite to the resultant of the equal parallel forces P.

165. *To find the ratio of the power and resistance in a system of pulleys, in which each pulley is supported by a separate string one end being fastened to the block of the pulley above it and the other end to a fixed beam, and all the strings being parallel.*



Let the weights of the pulleys be neglected.

The system is represented in the figure in which there are three moveable pulleys, and one fixed pulley.

Suppose in general there are n moveable pulleys and therefore n strings. Let the tensions of the strings be t_n, t_{n-1} .... t_2, t_1, reckoning from the lowest.

The equal parallel forces t_n support the resistance W, hence, $2t_n = W$. The tension of the highest string is equal to the power P: hence, we have

$$t_n = \frac{W}{2}$$

$$t_{n-1} = \frac{t_n}{2} = \frac{W}{2^2}$$

$$t_{n-2} = \frac{t_{n-1}}{2} = \frac{W}{2^3}$$

and so on: finally $t_1 = \frac{W}{2^n}$, and $t_1 = P$; therefore

$$\frac{W}{P} = 2^n.$$

Obs.—This is generally called First System of Pulleys.

166. *To find the relation between the power and resistance in the first system of pulleys when the weights of the pulleys are taken into account.*

Let the weights of the pulleys be w_1, w_2, w_3,...w_n; these forces and the resistance W are balanced by the power P. Let P_1 be the part of P which supports W; p_1 the part which supports w_1; p_2 the part which supports w_2 ....; p_n the part which supports w_n.

Then by the last case we have

$$p_1 = \frac{w_1}{2},\ p_2 = \frac{w_2}{2^2},\ \dots\dots\dots\ p_n = \frac{w_n}{2^n},\ P_1 = \frac{W}{2^n};$$

and
$$P = p_1 + p_2 + p_3 \dots\dots\dots\dots + p_n + P_1$$

Therefore
$$P = \frac{1}{2^n}\left\{W + w_n + 2w_{n-1} \dots\dots + 2^{n-1}w_1\right\}$$

This is the relation between P and W.

Cor.—When $w_1 = w_2 = \dots\dots = w_n = w$

$$P = \frac{1}{2^n}\left\{W + w\ (1 + 2 + 2^2 + \dots\dots + 2^{n-1}\right\}$$

$$= \frac{1}{2^n}\left\{W + w\,(2^n - 1)\right\}$$

or
$$W - w = 2^n\ (P - w).$$

167. *To find the ratio of the power to the weight in the first system of pulleys, when the strings are not parallel, the weights of the pulleys being neglected.*

Let $2\alpha_1$, $2\alpha_2$, $2\alpha_3$ .... $2\alpha_n$ be the angles between the portions of the strings separated by the first, second, third .... n^{th} pulleys, respectively. Then by Art. 164

$$t_n = \frac{W}{2 \cos \alpha_n}$$

$$t_{n-1} = \frac{t_n}{2 \cos \alpha_{n-1}} = \frac{W}{2^2 \cos \alpha_n \cos \alpha_{n-1}}$$

And so on: finally $t_1 = P$, and

$$\frac{W}{P} = 2^n \cos \alpha_1 \cos \alpha_2 \ldots\ldots\ldots\ldots \cos \alpha_n.$$

168. *To find the ratio of the power and resistance in a system of pulleys where the same string passes round all the pulleys.*

The system will be easily understood from the figure. There are two blocks; the lower one moveable and the upper one fixed; each block contains the same number of pulleys. The axes of the pulleys in each block may be arranged as in the figure, or they may be coincident. The same string passes round all the pulleys.

To find the relation between the power and resistance, it is only necessary to count the number of the parallel portions of the string which support the resistance, since the tension of the string is the same throughout.

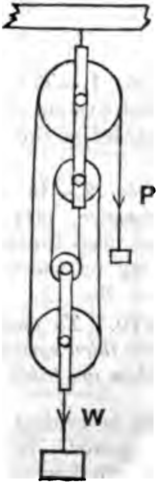
Thus if the number of strings at the lower block be n, we have

$$W = nP$$

If the weight B of the lower block be taken into account

$$W + B = nP$$

Obs.—This is generally called the Second System of Pulleys.



169. *To find the relation between the power and resistance in a system of pulleys in which there are* n *strings, all the strings being attached to a beam which supports the resistance.*

In the system there are n strings and $n-1$ moveable pulleys.

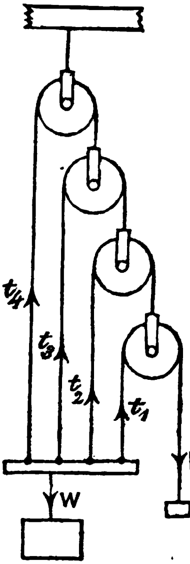
First, let the weights of the moveable pulleys be neglected.

Let the tensions of the strings be $t_1, t_2, t_3 \ldots\ldots\ldots$

Then $t_1 = P$; $t_2 = 2t_1 = 2P$; $t_3 = 2t_2 = 2^2P$; .... $t_n = 2^{n-1}P$.

And since all the parallel strings support W, we have

$$W = P(1+2+2^2+ \ldots\ldots\ldots +2^{n-1})$$
$$= P(2^n-1)$$
$$\frac{W}{P} = 2^n - 1$$



Obs. 1.—This is generally called the Third System of Pulleys. It may be regarded as the first system reversed.

Obs. 2.—In order that the beam, which supports the resistance, may be horizontal, the line of action of W must pass through the centre of the parallel forces $t_1, t_2 \ldots$ acting on the beam.

170. *To find the relation between the power and resistance in the third system of pulleys when the weights of the moveable pulleys are taken into account.*

It is evident that the weights of the pulleys will help the power. Let them be $w_1, w_2, w_3, \ldots\ w_{n-1}$ reckoning from the lowest.

Each of these forces supports a portion of W. Let the portion of W which P supports be W', and let the portions supported by the weights of the pulleys be $q_1, q_2, q_3 \cdots (q_{n-1})$. Then

$$W = W' + q_1 + q_2 + q_3 + \cdots\cdots + q_{n-1}.$$

also by the first case (Art. 169),

$$W' = (2^n - 1)P;$$

$$q_1 = (2^{n-1} - 1)\,w_1;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$q_{n-1} = (2-1)w_{n-1}.$$

Substituting these values

$$W = (2^n - 1)P + (2^n - 1)\,w_1 + \cdots\cdots + (2-1)\,w_{n-1}.$$

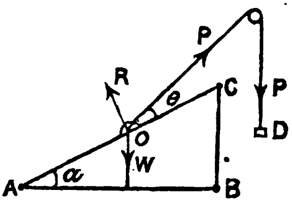
Cor.—If the weights of the pulleys be all equal, let

$$w_1 = w_2 = w_3 = \cdots\cdots\cdots = w$$

$$\begin{aligned} W &= (2^n - 1)\,P + w\left\{2^{n-1} + 2^{n-2} + \cdots\cdots + 2 - (n-1)\right\} \\ &= (2^n - 1)\,P + w\,\left\{2(2^{n-1} - 1) - (n-1)\right\} \\ &= (2^n - 1)\,P + (2^n - n - 1)\,w. \end{aligned}$$

or $$W + n\,w = (P + w)\,(2^n - 1).$$

171. THE INCLINED PLANE.—*A body is supported on a smooth inclined plane by a given force; to determine the conditions of equilibrium and the pressure on the plane.*



The diagram represents a section of the inclined plane by a vertical plane perpendicular to it; the power is represented as applied by means of a string passing over a pulley and supporting a body, D, whose weight is P, hanging freely.

Let θ be the angle which the string makes with the plane; W the weight of the body on the plane; α the inclination of the plane to the horizon.

The body O is kept in equilibrium by three forces P, W and the normal reaction R, of the plane. Hence, for equilibrium, these forces must lie in the same plane, and meet at a point. The direction of the string must then lie in a vertical plane through the centre of gravity of the body, O, perpendicular to the inclined plane, as represented in the diagram.

We have for equilibrium (Art. 41)

$$\frac{P}{\sin(RW)} = \frac{W}{\sin(RP)} = \frac{R}{\sin(PW)}$$

or

$$\frac{P}{\sin\alpha} = \frac{W}{\cos\theta} = \frac{R}{\cos(\alpha+\theta)}$$

These equations determine the ratio of P to W and the pressure on the plane, which is equal and opposite to R.

Otherwise.—Resolve the forces acting on the body O, parallel and perpendicular to the inclined plane (Art. 57)

$$P\cos\theta = W\sin\alpha \text{ .... } (1)$$

$$R + P\sin\theta = W\cos\alpha \text{ ... } (2)$$

Substituting the value of P, from (1), in (2) and solving for R, we get the same value as above. Equation (1) is the condition for equilibrium, (2) gives the pressure on the plane.

Obs.—Considering the inclined plane as a machine, P is the power and W the resistance. The hypotenuse of the right-angled triangle ACB is called the length of the plane, AB is called its base and BC its height.

172. Special Cases.—When the power acts parallel to the inclined plane $\theta = 0$, and hence from the formula of the last article

$$P = W\sin\alpha$$

or, *the power is to the resistance as the height of the plane is to its length.*

When the power acts parallel to the horizon $\theta = -\alpha$; hence

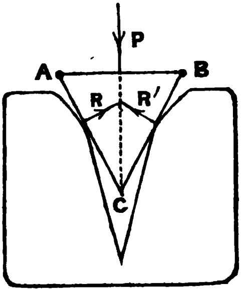
$$P = W\tan\alpha$$

or, *the power is to the resistance as the height of the plane is to its base.*

Obs.—The student should work out these special cases directly, by the method of Art. 41 and by resolving forces.

173. THE WEDGE.—The wedge is a solid triangular prism, and is used to separate obstacles by introducing its edge between them, and then forcing the wedge forward.

The diagram represents a section of the wedge, which is supposed to be isosceles, perpendicular to its axis.



Let P denote the force acting perpendicularly to face AB; R and R' the perpendicular reactions of the obstacle on the faces AC and BC; let the angle $ACB = 2a$.

For equilibrium these three forces must lie in one plane and meet at a point. Hence (Art. 41)

$$\frac{P}{\sin 2a} = \frac{R}{\cos a} = \frac{R'}{\cos a};$$

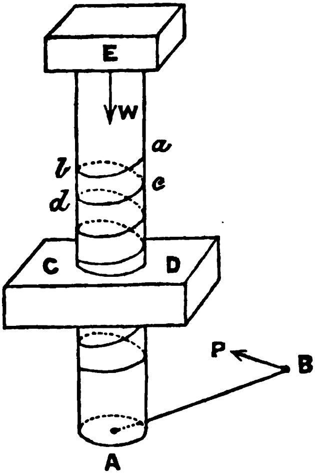
$$\therefore R = R',$$

and

$$P = 2R \sin a.$$

Obs.—The action of the wedge is essentially kinetical, and the statical condition is of little practical importance. Hatchets, nails, knives .... are modifications of the wedge.

173. THE SCREW.—The screw consists of a right circular cylinder AE with a uniform, projecting, spiral thread $a\ b\ c\ d$ .... running round its surface so as always to make a constant angle with lines parallel to the axis, AE, of the cylinder; the complement of this angle, which is the angle which the direction of the thread at any point makes with a plane perpendicular to the axis of the cylinder, is called the *pitch* of the screw.



The thread on the solid cylinder fits into an exactly similar groove in a hollow cylinder of equal radius cut out of a block *CD*. The block, when small, is sometimes called the *nut*, and the solid and hollow screws are called *companion screws*.

When in work, one of them is fixed, and the other is turned by means of a lever, *AB*, fixed in the cylinder at right angles to its axis. The lever is called the *power arm*. On turning the lever one of the cylinders must move backwards or forwards, by turning round its axis, and in this manner pressure may be exerted, or a body may be raised vertically against the force of gravity.

Obs.—The geometrical curve which the thread of the screw determines on the surface of the cylinder is called a *helix*. It may be generated by wrapping a piece of paper, cut in the form of a right-angled triangle, round a right cylinder. The base of the triangle being held perpendicular to the axis of the cylinder, the hypotenuse will trace a helix on its surface, whose *pitch* is the angle which the hypotenuse makes with the base.

175. *To find the ratio of the power to the resistance in the screw, when there is equilibrium.*

We shall consider the equilibrium of the solid screw supposed to be vertical. Let W be the weight of the solid cylinder AE and of the body E; P the power acting in a horizontal plane perpendicularly to the arm of the lever AB; α the pitch of the thread.

The equilibrium is maintained by the forces P and W and the reactions of the groove in the fixed block DC on the thread of the solid cylinder. These reactions are indefinite in number, but they all act in directions perpendicular to the thread, and therefore their lines of action make an angle α with lines parallel to the axis of the cylinder.

Let R be any one of these reactions; replace R by two components $R\cos\alpha$ acting vertically and $R\sin\alpha$ acting horizontally at right angles to the radius of the cylinder. Let the radius of the cylinder be a and the length of the power-arm AB equal to b. For equilibrium resolve all the forces vertically and take moments round the axis AE (Art. 93)

$$W = \Sigma\, R \cos \alpha \ldots\ldots\ldots\ldots (1)$$

$$Pb = \Sigma\, (a\, R \sin \alpha) \ldots\ldots\ldots (2)$$

Since a and α are constants, we get from (1) and (2)

$$\frac{W}{P} = \frac{b \cos \alpha\, \Sigma.R}{a \sin \alpha\, \Sigma.R}$$

$$= \frac{2\pi b}{2\pi a \tan \alpha}$$

$$= \frac{\text{circumference of circle described by power}}{\text{vertical interval between two threads}}.$$

Obs. 1.—Let $C =$ circumference of circle whose radius is AB, I the interval between two successive winds of the thread, measured parallel to the axis, the condition of equilibrium may be written

$$W I = P\, C.$$

Obs. 2.—The screw may be considered as a combination of a moveable inclined plane and a lever.

EXAMPLES.

1. Let the resistance caused by a nail fastened in a board be 224 lbs.; let the lever used to draw it be a hammer, which measures 2 in. from the point where the claw grasps the nail's head to the fulcrum; and let the handle be 13 in. long; find the force to be applied to the extremity of the handle which will just balance the resistance.

Ans. 34·46lbs.

2. In a lever of the second order, let the power be 10lbs., the resistance 12lbs., and the angle between their directions 120°; find the pressure on the fulcrum.

Ans. 11·13

3. A heavy body is suspended from the two ends of a straight lever without weight, whose length is 5 feet, by strings whose lengths are 3 and 4 feet; find the position of the fulcrum that the lever may rest in a horizontal position.

Ans. At a distance $3\frac{1}{5}$ ft. from the end to which the longer string is fastened.

4. The arms of a balance are equal, but one scale is loaded; find the true weight of a body in terms of its apparent weights when suspended at each end in succession.

Ans. The *A. M.* of apparent weights.

5. If one arm of a common balance be longer than the other, shew that the real weight of a body is the geometric mean between its apparent weights when suspended at each end in succession.

6. A body weighed in one scale of a false balance, the lengths of whose arms are a and b, appears to weigh W_1, and weighed in the other it appears to weigh W_2; find its real weight, W, and the weight, w, of each of the scale-pans, the C.G. of the beam being vertically below the fulcrum.

Ans. $$W = \frac{aW_2 + bW_1}{a+b};$$

$$w = \frac{W^2 - W_1 W_2}{W_1 + W_2 - 2W}.$$

7. The arms of a false balance are unequal, and one of the scales is loaded; a body whose true weight is P appears to weigh W when placed in the loaded scale, and W' when placed in the other scale; find the ratio of the arms and the weight of the mass with which the scale is loaded.

Ans. $$\frac{P - W'}{W - P};\quad \frac{P^2 - WW'}{W - P}.$$

8. In the Danish steelyard, if a_n be the distance of the fulcrum from the end where the scale-pan is attached, when the mass of the substance in the pan is n lbs., prove that

$$\frac{1}{a_n} + \frac{1}{a_{n+2}} = \frac{2}{a_{n+1}};$$

9. In a wheel and axle the diameter of the wheel is 1 yard and the diameter of axle 6 inches; the string on the wheel is just strong enough to support a tension of 42lbs.; find the greatest weight which can be supported.

Ans. 252lbs.

10. An endless string passing over a fixed pulley is just strong enough to support another heavy pulley of equal radius when the portions of the string between the pulleys are parallel; show that it will break if the portions of the string cross each other.

11. In the first system of pulleys, if a power of 3lbs. support a resistance of 48lbs., how many moveable pulleys are there?

Ans. 4.

12. In the first system of pulleys, let there be 3 pulleys, the weight of each pulley being 2lbs.; find the resistance which may be supported by a power of 10lbs.

Ans. 66lbs.

13. In the second system of pulleys a platform is suspended from the lower block; find the force which a man, whose weight is W, standing on the platform must exert on the string in order to support himself, the number of strings at the lower block being n, and mW being the weight of platform and the lower block together.

Ans. $\dfrac{m+1}{n+1} W$.

14. In the third system of pulleys let there be 3 moveable pulleys, the weight of each being 1lb.; find the resistance which will be supported by a power of 9lbs.

Ans. 146lbs.

15. In the third system of pulleys if the tensions of the strings increase in geometrical progression, so must, also, the weights of the pulleys.

16. In the first system of pulleys, if each of the strings, instead of being attached to the fixed beam, is made to pass over a fixed pulley and then attached to the upper part of the block of the pulley which it supports; find the relation between the power and resistance, (1) neglecting the weights of the pulleys, (2) when the weight of each pulley is w.

Ans. $W = 3^n P$;

$2W - w = 3^n (2P - w)$.

17. In the third system of pulleys, if each of the strings, instead of being attached to the moveable beam, is made to pass under a pulley fixed to the beam, and then attached to the lower part of the block of its pulley, find the relation between the power and resistance, (1) neglecting the weights of pulleys, (2) when the weight of each pulley is w.

Ans. $W \qquad P$;

$2(W+nw) = ($

18. In the third system of pulleys let there be n strings; find the distance from the first string at which the resistance must act in order that the beam may be horizontal, the radius of each pulley being r and their weights being neglected.

Ans. $\frac{(n-2)\,2^n+2}{2^n-1}\,r$.

19. An inclined plane rises 3 feet 6 inches for every 5 feet of length; what force must be exerted parallel to the plane in order to prevent a body, whose weight is 200lbs., from slipping down?

Ans. 140lbs.

20. A body, whose weight is W, may be supported on a certain inclined plane by a power, P, acting horizontally, or by a power, Q, acting parallel to the plane; show that

$$\frac{1}{Q^2}=\frac{1}{P^2}+\frac{1}{W^2}.$$

21. If R and R' be the pressures on an inclined plane when a body of given weight, W, is supported by a power parallel to the base and a power parallel to the plane, respectively,

$$RR'=W^2.$$

22. When a body of given weight is supported on a smooth inclined plane by a given power, prove that there are generally two directions in which the power may act, and that the sum of the pressures on the plane in these cases is double the pressure when the power is the least possible.

23. If there be 5 turns of a screw in one inch, and if the power-arm be one foot in length, find what resistance will be balanced by a power of 10lbs.

Ans. 3769·9lbs.

24. If the tangent of the pitch of a screw be $\frac{1}{8}$, the radius of the cylinder 2 inches and the length of the power-arm 1 yard; find the mechanical advantage.

Ans. 144.

25. If a screw make m turns in a cylinder one foot long, and the length of the power-arm be n feet, find the mechanical advantage.

Ans. $2\pi m n$.

26. An endless string hangs at rest over two pegs in the same horizontal plane, with a heavy pulley in each festoon of the string; if the weight of one pulley be double that of the other, show that the angle between the portions of the upper festoon must be greater than 120°.

CHAPTER VIII.

WORK AND ENERGY.

176. In this chapter a short explanation will be given of the sense in which the terms *Work* and *Energy* are used in science. The Simple Machines will furnish illustrations of certain cases of the transformation of energy, and the subject will be further briefly considered in Kinetics. The beginner, however, should understand that no adequate conception can be formed of the universality of "the Principle of the Conservation of Energy" without a tolerably wide acquaintance with Physical Science and Chemistry. The illustrations given of the principle in this book will, it is hoped, assist in giving a clearer idea of the functions of a machine, and serve as an introduction to further studies on the subject of Energy.

177. Work.—When a body is moved against force of any kind *work* is said to be done against the force. For example, when a mass is lifted vertically work is done against the force of gravity.

Let the weight of the mass be W, and h the vertical distance through which it is raised. Then the work done against W through the space h is directly proportional to W and also to h, thus

$$\text{work done} = kWh,$$

where k is the constant of variation whose value depends on the unit of work chosen. If we choose for the unit of work the work done when $W = 1$ and $h = 1$, we have

$$\text{work done} = Wh.$$

This is the work done against the force W through a vertical space h, whatever be the agency by which the work is performed, and whatever be the time of performance.

For practical measurements of work the unit of force used is the gravitation unit. The British unit of work is the work done in overcoming a force equal to the weight of a pound through the space of one foot, and is called a Foot-Pound. The international unit of work is the Kilogramme-Metre or the Sér-Metre.

For scientific measurements, when accuracy is required, the unit of work is the work required to overcome the kinetic unit of force through the unit of space.

Engineers call the rate at which an agent works when overcoming a resistance equal to 33,000 times the weight of a pound through a space of one foot in a minute a Horse-Power.

178. **Work done by a Force.**—Let A be the point of application of a force, P, acting on a body. Suppose A at first coincides with a fixed point O in space, and let OB be the line in which the force acts.

Suppose now that the body moves so that A moves from O, along OB, through any space a, while the force P continues to act with uniform intensity, then

$$\text{work done by } P = Pa.$$

When A moves along BO, in the direction opposite to P, through any space a,

$$\text{work done against } P = Pa.$$

When A moves through a space a, along a line OC making an angle α with OB, while P continues to act at A parallel to its original direction,

$$\text{work done by } P = Pa \cos \alpha.$$

For, P may be resolved into two components $P\cos\alpha$ along OC, and $P\sin\alpha$ perpendicular to OC; the latter component cannot affect the motion along OC, and hence can do no work.

The above expression includes the case when work is done *against* a force, if such work be considered as negative work done *by* the force; for, when work is done against a force the angle which the direction of the force makes with the line of displacement of its point of application is greater than a right angle and hence its cosine is negative.

We have supposed P to continue to act parallel to OB during the displacement of its point of application. This may always be considered to be the case if the displacement

a be taken indefinitely small. Thus, in general, *the work done, during an indefinitely small displacement of the point of application of a force, is the product of the resolved part of the force in the direction of the displacement into the displacement.*

Obs. 1.—Since $Pa\cos\alpha$ is the work done by P for a displacement a of its point of application, $P\cos\alpha$ is the work which would be done for a unit of displacement. Hence the resolved component of a force in any direction is equal to the space rate at which the force would commence to work, if its point of application moved in the given direction. It should be observed that the work done by a force does not depend on the time of motion, or velocity, of its point of application.

Obs. 2.—$a\cos\alpha$ has been called the *Virtual Velocity*, and $P\,a\cos\alpha$ the *Virtual Moment*, of the force P, when the displacement is indefinitely small.

179. ENERGY.—The capacity to perform work in moving matter against force is called *energy*.

A body possesses energy in virtue of its position with respect to forces acting on it, and in virtue of its motion, which is change of position.

A body in visible motion possesses *kinetic energy;* that is, it is capable, in virtue of its motion, of performing a definite amount of work. It will be shown in Kinetics that this form of energy depends only on the mass of the body and the rate at which it is moving.

A body possesses *potential energy* in virtue of its position with respect to forces acting on it; that is the forces acting on the body have the power to perform a definite amount of work during a change of the body's position.

For example, when a body whose weight is W descends, in any manner, through a height h, work is done, by the force of gravity, equal to Wh. In this case the body is said to have lost an amount of *potential* energy measured by Wh.

In general, when the altitude of a body, whose weight is W, is changed by an amount h, its potential energy with reference to gravity is changed by an amount Wh.

180. THE LAW OF CONSERVATION OF ENERGY.—The following law is found to hold in all cases in Nature.—*The total amount of energy of any body, or system of bodies, can only be altered by external forces acting on the system, and the change of energy is equal to the work done by, or against, the external forces.*

The forces acting on a body may, in general, be divided into two classes: (1) moving forces which can produce as well as resist motion, (2) purely resisting forces which can only modify sensible motion, by changing it into molecular motion, when sensible motion is stopped or retarded. Art, 17. When a body is in motion under the action of such a system of forces work is done by any moving force which has a resolved component in the direction of the motion of its point of application, and work is done against any moving force which has a resolved component in the direction opposite to the motion of its point of application. The work done *against* the moving forces is equal to the gain of the body's potential energy with reference to these forces, the circumstances being supposed to be such that these forces would act equally and in the same directions if the motion of the body were reversed. Of work done against purely resisting forces we shall only have to mention work done against friction, and what becomes of the energy so spent will be stated further on. The law of energy in dynamics may be thus expressed.—*The work done in any time on a body by a system of forces is equal to the whole effect in the forms of potential and kinetic energy together with the work lost against friction.*

In what follows, except when the contrary is expressed, we shall suppose the body or system to be capable of moving without friction along the surfaces (when there are such) by which its motion is guided.

If we consider the *whole* potential energy of the body with reference to *all* the moving forces, resisting as well as working, the law of energy may be stated that *the sum of the potential and kinetic energies of the body is constant.*

Also, considering the work done against a force as negative, the *whole* work done during any motion, when positive, is equal to the increase of kinetic energy or the decrease of potential energy; and the *whole* work done during any motion, when negative, is equal to the increase of potential energy or the decrease of kinetic energy.

181. When a body under the action of given forces is in *neutral* equilibrium (Art. 110), that is, when it can move so that the forces acting on it shall still continue in equilibrium during the motion, the body when once put in motion will, according to the first law of motion (to be explained hereafter), move at a uniform rate, as long as the forces acting on it continue in equilibrium. In this case the kinetic energy of the body remains constant, because it depends only on the mass of the body and the rate of motion. During such a motion some of the forces do work on the body while work is done against others. Hence, according to the Law of Energy, the work done on the body by the one set of forces is equal to the potential energy gained by the body with respect to the other set. Or, considering work done against a force as negative work done by the force, *the whole work done by the forces on the body is equal to zero.*

Conversely, when during any motion of a body its kinetic energy remains constant the *whole* work done by the forces is equal to zero, and provided the only possible motion of the body is the one then being executed (or its opposite), the forces would be in equilibrium and the body would remain at rest in any position which it occupies during the motion, if then and there it could be deprived of its kinetic energy.

182. In general, however, when a body is in motion under the action of forces not in equilibrium, the whole work done on the body is not equal to zero and the body's kinetic energy changes, the amount of change during any change of position being (by the law of energy) equal to the *whole* work done on the body by the forces in consequence of the change of position.

In any position which the body occupies during the motion its kinetic energy must in general be changing at a definite space rate (depending on the way in which the body is moving with reference to the forces acting on it, not on the time of motion, or velocity). But if there are any positions at which the rate of change of the body's kinetic energy is zero (that is, at which the rate of change of the *whole* work done on the body is zero), at such a position the forces acting on the body would be in equilibrium, supposing the only possible motion of the body, from that position, to be the particular motion which the body is then executing (or its opposite), and, on this supposition, in any of these positions the body would remain at rest, if then and there it could be deprived of its kinetic energy.

In order then to determine whether in any particular position the forces which act on a body are in equilibrium or not, the whole work which would be done by the forces during any possible displacement of the body from that position, if the forces during the displacement be supposed to continue to act with equal intensities parallel to the directions which they have at the particular position under consideration, must be calculated. The whole work thus computed is proportional to the rate of change of the body's kinetic energy (no matter what be the magnitude of the hypothetical displacement) and if it vanishes for *all* such displacements the forces are in equilibrium, and not otherwise.

The condition that the forces shall continue to act with equal intensities parallel to the directions which they have at the position of the body in question may be secured by supposing any actual displacement of the body, which may be taken, to be indefinitely small. Hence—*When the whole work done by all the forces which act on a body (or system of bodies) during every possible indefinitely small displacement of the body is equal to zero, the forces are in equilibrium.* And conversely—*When the forces acting on a body are in equilibrium, the whole work done during any possible indefinitely small displacement of the body is equal to zero.*

Conditions sufficient for the equilibrium of forces acting on any body, or system of bodies, can always be obtained by equating to zero the whole work done by the forces during any indefinitely small displacements which can be made without breaking the connections of any of the parts of the system.

Obs.—The proposition of this article, which is celebrated in the history of Dynamics, has been called *the Principle of Virtual Velocities* and may be thus enunciated—*When the algebraical sum of the virtual moments of a system of forces is zero for every displacement, the forces are in equilibrium.* And conversely—*When a system of forces is in equilibrium, the algebraic sum of their virtual moments is zero for any displacement.*

183. In the last chapter we considered machines in equilibrium, and a machine was defined as an instrument for the transmission and alteration of force. A machine may be more correctly defined as an instrument, or system of bodies, for the purpose of transforming energy from a less convenient to a more convenient kind, and the function of a machine is to perform work against some external force. We shall, as before, suppose the parts of a machine to be capable of moving freely on each other without friction, in other words that there is no resistance to motion except the external resistance.

When a machine is in uniform motion work is done by the power, and this work is transformed into work done against the resistance, and the relation between these forces may be calculated from the Law of Energy, which in this case may be stated.—*The work done by the power is equal to the work done against the resistance.* This is sometimes called the "principle of constancy of work done."

When the power is the weight of a given body, and the resistance to be overcome the weight of another body, the work done by the power is equal to the potential energy, with reference to gravity, lost by the first body; and the work done against the resistance is equal to the potential energy gained by the second. Thus, the potential energy lost by one is gained by the other, and the whole potential energy of the system remains constant.

When the equilibrium of a machine is not neutral, its motion cannot be uniform, and there is not equality between the work done by the power and the work done against the resistance, during any motion of the machine. Part of the potential energy is in fact transformed into the kinetic energy of motion, which changes because the rate of motion changes.

The relation between the power and resistance, necessary and sufficient for equilibrium, may, however, in all cases, be determined by equating the work done by the power, during an indefinitely small displacement of its point of application, (supposed to be made in such a manner as not to break the connections of any of the parts of the system) to the work done against the resistance, during the corresponding necessary displacement of its point of application.

Obs.—In the case of a machine moving uniformly, we may suppose the machine to have been put in motion, at first, by some external agency different from the power: the machine would continue to move with kinetic energy equal to the energy spent in putting it in motion. Or, we may suppose the power at first to have been greater than sufficient to balance the resistance; thus the excess of work done at first by the power, over the work done against the resistance, would be transformed into kinetic energy, and if the excess of power were removed the motion would continue uniform from that instant.

184. We now proceed to show how the relation between P and W, in the Mechanical Powers, may be deduced from the Law of Energy, when the machine is at rest (or when moving uniformly) under the action of these forces; and, conversely, how assuming the relation between P and W for equilibrium (or uniform motion) the Law of Energy may be shown to hold.

185. THE LEVER.—Suppose the lever to turn in the plane of the forces, about the fulcrum, through an indefinitely small angle (circular measure θ) so that the points of application of P and W describe indefinitely small arcs,

which may thus be considered as coinciding with their lines of action (Figs. Art. 155).

$$\text{work done by } P = P.FA.\theta$$

$$\text{work done against } W = W.FB.\theta$$

therefore $$P.FA.\theta = W.FB.\theta$$

hence $$P.FA = W.FB$$

which is the condition of equilibrium.

Conversely, assuming the condition of equilibrium, it may be shown that the work done by P, during the indefinitely small displacement, is equal to the work done against W.

186. THE WHEEL AND AXLE.—We may suppose the machine to be in a state of uniform motion, and it is evident that while the point of application of P descends through a height equal to the circumference of the wheel, the point of application of W is raised through a height equal to the circumference of the axle (Fig. Art. 162).

Hence $$\text{work done by } P = P \times \text{cir. of wheel};$$

$$\text{work done against } W = W \times \text{cir. of axle};$$

therefore $$P \times \text{cir. of wheel} = W \times \text{cir. of axle},$$

which is the condition of equilibrium (or of uniform motion).

Conversely, assuming the condition of equilibrium, we may show that the work done by P is equal to the work done against W.

Obs.—It will be seen that the loss of potential energy with reference to gravity in the mass which descends is equal to the potential energy gained by the mass which is lifted, and thus the potential energy of the system as a whole remains unaltered. If we consider P as an external force acting on the system consisting of the machine and the mass to be lifted, the work done by P is transformed into the gain of potential energy of the mass lifted.

187. THE FIRST SYSTEM OF PULLEYS.—Let the weights of the moveable pulleys be neglected. Suppose the point of application of P to descend through a height h (Fig. Art. 165); this diminishes the lengths of the parallel portions of the

When, however, the power acts parallel to the plane, the equilibrium is neutral. This is the most important case in practice, and it will be seen that while the body D descends through a space equal to the length of the plane, the body O is raised through a vertical distance equal to the height of the plane. Thus P is to W as the height of the plane is to its length.

191. THE SCREW.—It is evident that while the point of application of P describes the circumference, C, of the Power circle, the point of application of W is moved through a space equal to the interval, I, between two successive threads. Hence

$$WI = PC.$$

Obs.—The student will have no difficulty in seeing that the work done by P during one revolution is PC. For P is supposed to act always perpendicularly to the power-arm, and the circumference of the circle may be considered as the perimeter of a regular polygon having an infinite number of sides.

192. *To deduce the condition of uniform motion on a rough inclined plane, when the power acts parallel to the plane.*

Let μ be the coefficient of kinetic friction. Resolve W into two components $W \sin \alpha$, parallel, and $W \cos \alpha$, perpendicular, to the plane. The force of resistance to motion up the plane is $W \sin \alpha + \mu W \cos \alpha$ (Art. 144). Hence if the body move up the plane through a space s

$$\text{work done by } P = P\,s$$

$$\text{work done against resistance} = (W \sin \alpha + \mu W \cos \alpha)\,s$$

$$\text{therefore} \qquad P = W \sin \alpha + \mu W \cos \alpha.$$

Obs.—The work done by P is not in this case equal to the potential energy gained by the body O (Fig. Art. 171). Part of the work done is spent in overcoming friction and is transformed into *heat*, a form of molecular energy. The quantity of energy so spent during the motion through the space s is $\mu Ws \cos \alpha$.

193. It will be now understood that however the application of a force be changed by the intervention of a machine, in no case can more work be done against W than is done by P; and in practice as it is impossible to construct any machine so as to move without friction, the work performed is always less than the energy applied. The ratio of the work done by a machine to the work applied is called the *efficiency* of the machine.

In Art. 152, the so called "mechanical advantage" of a machine was defined to be the ratio of W to P. It has been shown in this chapter that the advantage gained (when there is advantage) consists in the transformation of energy from a less convenient to what may be, according to circumstances, a more convenient kind; so that in the same proportion that W is greater than P, the space through which P acts is greater than the space through which W is overcome. The principle is commonly stated thus: ***what is gained in power is lost in speed.***

There are, however, many cases in which advantage is derived more from speed than power. For example, in the muscular mechanism of animals, levers of the third class (in which there cannot be mechanical advantage in the sense spoken of in Art. 152) are employed by Nature in turning the limbs about the joints. In these, and such, cases the loss in power is compensated for by a gain in rapidity of action.

194. NATURE OF EQUILIBRIUM.—The theory of energy furnishes a simple test for determining whether the equilibrium of a body is ***stable***, ***unstable***, or ***neutral*** in any case.

Suppose a body to be in equilibrium under the action of given moving forces, and the reactions of smooth immoveable surfaces, only.

When just as much work would be done *by* as *against* the moving forces in any possible displacement of the body, which does not break the connection of its parts with one another, or with the frictionless constraints to which it is subject, the equilibrium is neutral.

For, suppose the body to be displaced from its position of equilibrium, by some external agency which imparts to it kinetic energy, whose amount may be as small as we please. Since during the displacement the whole work done on the body by the moving forces acting on it is zero (the reactions of the smooth surfaces of course do no work, Art. 178) the kinetic energy remains constant. If we suppose the kinetic energy of the body to be now taken away by some external agency, so that the motion of the body is stopped, the body will remain at rest in the new position and hence the original position is neutral. Art. 110.

In a similar manner it may be shown, (since the moving forces are supposed to be such that they would act equally and in the same directions if the motion of the body were reversed) that when in every small displacement (not infinitely small, see Art. 182) less work is done by than against the moving forces, the equilibrium is stable: and that when in any small displacement more work is done by than against the moving forces, the equilibrium is unstable. For, in the first case, the kinetic energy (supposed to be imparted to the body) would decrease and ultimately vanish, after which the body would return to, and oscillate about, its position of equilibrium; but in the second case the kinetic energy would increase, and the body would move further away from its position of equilibrium.

195. It will be understood from the above that the *whole* potential energy of the body with reference to all the moving forces must be constant for positions of neutral equilibrium; that it increases when the body is displaced in any way from a position of stable equilibrium, and hence at such a position is a minimum; and that it decreases when the body is displaced in any way from a position of unstable equilibrium, and hence at such a position is a maximum.

It also follows from the nature of maxima and minima values, that, conversely, the body will be in equilibrium when placed in positions (without disturbing the connections of the parts) for which its whole potential energy is a maximum or a minimum.

For, during an infinitely small displacement from such a position the space rate of change of the *whole* potential energy is zero, and hence the position is one of equilibrium. Art. 182. This, however, introduces a conception with which the student is at present not supposed to be familiar.

196. As a particular case of the theory of the last article we may consider a body, or system of bodies, acted on only by gravity and the reactions of smooth immoveable surfaces.

When the body is placed so that its centre of gravity is at the lowest position which it can attain by moving the body, consistently with the connection of its parts with one another and with the frictionless constraints, it is in stable equilibrium. When the centre of gravity is at its highest position the body is in unstable equilibrium; and when the altitude of the centre of gravity of the body remains the same during any such motion of the body, any position is one of neutral equilibrium.

For example, let us take the case of a uniform beam resting between two smooth inclined planes at right angles to each other. Fig. Art. 139. Let G be the centre of gravity of the beam and C the intersection of the planes. Then it is evident, geometrically, that the distance CG is constant, so that in all displacements (in a vertical plane perpendicular to the line of intersection of the inclined planes) G must move on a circle whose centre is C. If the beam be so placed that G shall coincide with the highest point of this circle, the beam will be in a position of unstable equilibrium.

197. OTHER FORMS OF ENERGY.—A discussion of the different forms of energy is unsuited to the limits of this treatise; but it may be mentioned, for information, that matter not only possesses energy in virtue of its visible position with respect to force, and its visible motion, but also in virtue of its molecular condition and molecular invisible motion.

All forms of energy may be divided into two classes—*visible energy* and *molecular energy*. Matter in virtue of its molecular state possesses energy which may be kinetic or potential; that is, it is capable of performing a definite amount of work, which can be calculated, and in many cases *has* been expressed in foot-pounds. Heat, light, electricity, magnetism and chemical action are forms of the molecular energy of matter. For example, the work done by a steam engine is derived from heat, which is a form of molecular kinetic energy; this energy, again, is derived from the molecular potential energy of the carbon and hydrogen of the fuel with respect to the force of chemical affinity of the oxygen of the atmosphere; and the potential energy of the fuel is the result of transformation in vegetables of the molecular kinetic energy of solar radiation.

In speaking of machines we have supposed the whole of the energy applied to be transformed into work, or work and visible kinetic energy. This is never practically the case on account of friction and other resisting forces. Part of the applied energy is always transformed into molecular energy. For example, the energy required to overcome friction is transformed into heat, which always accompanies friction.

It was formerly supposed that energy was *absolutely* lost against friction and other purely resisting forces. Modern experimental enquiries, however, have demonstrated that energy is as indestructible as matter; it can neither be created nor destroyed, but only transformed from one form to another. This grand generalization of modern science is enunciated in the law of the conservation of energy.—*The whole amount of energy in the universe, or in any limited system which does not receive energy from without, or part with it to external matter, is invariable.*

EXAMPLES.

1. Calculate in foot-pounds the work done in raising a body whose mass is 35lbs. through 6 feet vertically.

Ans. 210.

2. What is the work done in the same case, in kinetic units of work, at a place where $g. = 32{\cdot}15$?

Ans. 6751·5.

3. Calculate, in kinetic units of work, the work done in raising a body whose mass is 14lbs. through a vertical height of 10 feet, at Calcutta ($g. = 30{\cdot}112$.)

Ans. 4495·7.

4. In the last example, what would be the work done in lifting the same body at Edinburgh ($g. = 32{\cdot}2$)?

Ans. 4508.

5. Show that the work done in raising a body whose mass is 14lbs. through a height of 40 feet is the same as the work done in lifting a mass of 20lbs. through a height of 28 feet.

6. When bodies are raised vertically through various heights, show that the whole work done against gravity is equal to the work which would be done in lifting a mass equal to the sum of the masses of the bodies, from the first position of the centre of gravity of the system to the last.

7. In the wheel and axle, supposing the power to be the weight of a body which descends, and the resistance the weight of a body which ascends, show that when there is uniform motion the altitude of the common centre of gravity remains invariable.

8. On the same supposition, as in the last question, show that a similar proposition is true for the different systems of pulleys, and for the inclined plane, when the power acts parallel to the plane.

9. Show that the work done in dragging a body along a smooth inclined plane is equal to the work done in raising the body through the height of the plane.

10. Prove that the work done in dragging a body along a rough inclined plane is equal to the work done in dragging the same body along the base of the plane (supposed rough) together with the work done in lifting the body through the height of the plane.

11. If the force required to draw a load along a level road be the one-fortieth part of the weight, what is the work done in drawing a load of 1 ton up a hill 50 feet long, which rises 3 feet in every 5?

Ans. 69440 foot-pounds.

12. A beam, whose centre of gravity is at a given distance from its middle point, rests inside a smooth hemi-spherical bowl; show that its centre of gravity is vertically below the centre of the sphere and that the equilibrium is stable.

13. Two unequal particles, connected by a string, support each other on the convex arc of a smooth vertical circle; show that the common centre of gravity is in the vertical through the centre of the circle, and that the equilibrium is unstable.

KINEMATICS.

CHAPTER I.

VELOCITY.

1. **Kinematics** is the science which treats of the motion of geometrical figures—points, lines, curves, surfaces and solids. It involves the idea of time as well as space, and in this it differs from Geometry which is concerned with space notions only.

We propose to consider only such questions in Kinematics as will afterwards be useful in the Kinetics of a particle, (and of a rigid body, when its motion may be represented by that of a particle) and it will be only necessary to investigate some simple cases of the Kinematics of a point and a right line.

The motion of a point will be supposed to be regulated by certain given conditions, or laws, in accordance with which it traces out, continuously, a certain straight line (or lines), or a curve which is called its *path* or *locus*, and only cases in which the point moves in one plane will be considered.

In the Kinematics of a point we have to investigate the rate of motion, the direction of motion and the position of the point at any instant; the time occupied in passing from one position to another, and the geometrical nature of the path or locus which the point traces out.

Obs.—In what follows the word *space* is used as an abbreviation for *length of path.*

2. **Velocity.**—The rate at which a point is moving at any instant is called its *velocity* at that instant. When a point passes over equal spaces (lengths of path) in equal intervals of time, however large or small the intervals be taken,

its velocity is said to be *uniform*, and is the same at every instant; when this is not the case the velocity is *variable.*

The most convenient unit of velocity, and that which is generally used, is the velocity of a point which passes over the unit of space uniformly in the unit of time.

Uniform velocity is measured by the number of units of space passed over in the unit of time. The unit of length is usually taken to be a foot, and the unit of time a second, and velocity is thus expressed in feet per second. For example, with these units of length and time, the velocity of a point which passes over one foot uniformly in one second is the unit of velocity, and the velocity of a point which passes over 10 feet uniformly in one second is 10 units of velocity, or a velocity of 10 feet per second.

3. *A point moves with a uniform velocity* v *over any path; required to find the space* s *passed over in a given time* t.

By the method of measuring velocity, the number of units of space passed over in the unit of time is v. Hence, since the point moves uniformly, vt is the number of units of space passed over in the time t, thus—

$$s = vt$$

Cor.—Let a be the distance (measured along the path) of the moving point, at any instant, from a fixed point on the path, and s the distance at the end of a time t, then

$$s = a + vt$$

where v is to be taken positive or negative according as the distance s increases or decreases with the time.

4. In order to explain more clearly the method of measuring uniform velocity, and the reason for adopting the particular unit of velocity which has been chosen, we give another method of investigating the relation between time, space and velocity.

We may define uniform velocity as a quantity which is directly proportional to the space passed over in a given time, and inversely proportional to the time of passing over a given space. For, if one point, in a given time, passes over twice as great a space as a second point, it is said that the velocity of the first is twice as great as that of the second; and so on. Again, if the first point only takes half the time which the second takes to pass over a given space, we say that the velocity of the first is also twice as great as that of the second; and so on. Thus—

$$v \propto s \ldots\ldots\ldots \text{ when } t \text{ is constant,}$$

and $$v \propto \frac{1}{t} \ldots\ldots\ldots \text{ when } s \text{ is constant;}$$

therefore, when both s and t vary

$$v = k\frac{s}{t}$$

k being the constant of variation, the numerical value of which depends on the unit of velocity chosen, the units of time and space being supposed known.

If we take for unit of velocity the velocity of a point which passes over the unit of space uniformly in the unit of time, we have $v = 1$ when $s = 1$ and $t = 1$. Thus $k = 1$, and we get the convenient equation, connecting velocity, referred to the above unit, with space and time,

$$v = \frac{s}{t}$$

If we make $t = 1$ we get $v = s$; that is, v units of velocity are *numerically* equal to s units of space. Hence, the space passed over in the unit of time is a measure of the velocity of a point moving uniformly, and this is found to be the most convenient measure.

Velocity might also be measured by the reciprocal of the time occupied by a point in passing over the unit of space. For, if we make $s = 1$ we get $v = \frac{1}{t}$.

5. When a point moves uniformly its velocity is constant; that is, it is the same at every instant of the motion. Its numerical value may be derived from the formula

$$v = \frac{s}{t}$$

where s is the space actually passed over in *any* given time t. Thus we shall obtain the same result whether we derive v

from the space described in a million seconds, or from the space described in the one-millionth part of a second. The fraction, space passed over divided by the time of motion, expresses the space which *would be* described in a unit of time at the same rate, no matter how small the time be taken.

6. Variable Velocity.—When a point does not move uniformly, its velocity is different at different instants of time, but it is evident, however, that at any given instant the point must have a definite velocity, or rate of motion, and it is now to be explained how this velocity is estimated.

The velocity of a point when variable is measured at any instant by the velocity of a second point moving *uniformly* at the same rate which the first has at the instant under consideration.

Or thus—The velocity at any instant is measured by the space which would be passed over in a unit of time, supposing the rate of motion to continue uniform and the same as at the instant under consideration.

This statement is in fact nothing more than what the equation for uniform velocity

$$v = \frac{s}{t}$$

expresses when the velocity is derived from the space described in an indefinitely short time, Art. 5, during which time the motion may be considered uniform.

The velocity of a point may then always be derived from the above equation, but when the velocity is variable we must derive the velocity at any instant from the space passed over in an indefinitely small time reckoned from the instant.

Def.—The *average* velocity of a point during any time is the space described in that time divided by the time.

7. Illustrations.—The idea of a variable rate of motion is more or less familiar to every one, and the scientific method of measuring it, at any instant, is not different from that used in every day life. Suppose, for example, that we

travel in a railway carriage 30 miles between two places in an hour. We cannot say that the velocity of the train was 30 miles an hour at every instant during the journey (although this would be the average velocity for the hour) since at the commencement and end of the journey the velocity of the train was very much less, and at some intermediate periods must have been much greater, than 30 miles an hour, or the whole 30 miles could not have been passed over in the hour. We estimate the velocity of the train at any instant by saying that it is moving at a certain rate just then, although before it may have been moving, and afterwards may move, quicker or slower. Thus when we say at any time that the train is moving at a rate of 40 miles an hour, or has a velocity equal to 40, taking a mile and an hour as the units of space and time, we mean that if the train went on for an hour from that time *at the same rate* it *would* pass over 40 miles. Thus the velocity at any instant is not to be measured by the actual space which is passed over in the next unit of time, but by the space which would be passed over if the velocity did not change.

Again, when we speak of the velocity of a moving point being at any instant 32 feet per second, we do not mean that it has passed over 32 feet in the last second, nor that it will pass over 32 feet in the next second, but that its velocity at that particular instant is equal to that of a point which would move *uniformly* over 32 feet in one second.

8. RELATIVE VELOCITY.—When two points move along the same path the rate at which the space between them increases or decreases is called the *relative velocity* of one with regard to the other, estimated along their common path.

When each of the points moves uniformly the relative velocity is evidently uniform and is measured by the increase or decrease of the length of path between them in the unit of time. Hence, obviously, the relative velocity is equal to the sum or difference of the absolute velocities, estimated along the path, according as the points move in opposite directions or in the same direction.

Let s be the increase or decrease of the space between the points in the time t, and u the relative velocity ; then, as in Art. 3,

$$s = ut$$

Again, let a be the space between the points at any instant and s the space between them at a time t. Then, when the space decreases

$$s = a - ut.$$

The points will meet when $s = 0$ which gives for the time of meeting

$$t = \frac{a}{u}$$

When the space between the points increases

$$s = a + ut.$$

The time when the points met is found by making $s = 0$ to be

$$t = -\frac{a}{u}$$

the negative value for the time indicating that the event took place before the instant from which the time is reckoned.

Obs. 1.—The relative velocity may be taken positive or negative according as the space between the points increases or decreases.

Obs. 2.—It is evident that the relative motion of either of the points with respect to the other will remain unaltered if we suppose one of the points to be at rest and the other to move with a velocity equal to their relative velocity.

Obs. 3.—The relative velocity when variable is measured at any instant by what would be the increment of space between the points in a unit of time, if for such a period it were to keep its value unchanged.

9. **ANGULAR VELOCITY.**—The rate at which a straight line, moving in one plane, is changing its direction at any instant, is called its *angular velocity* at that instant.

The change of direction of the moving line may be measured by the angles which it makes with *any* fixed line in the plane of its motion, and angular velocity, when uniform and when variable, is estimated in exactly a similar manner to linear velocity.

Thus, the unit of angular velocity is the angular velocity of a line which describes the unit of angular space uniformly in the unit of time. Angular velocity, when uniform, is measured by the angle described in the unit of time, and when variable, it is measured at any instant by the angle which would be described in a unit of time, if for such a period it were to keep its value unchanged.

Again, let ω be the angular velocity of a line, when uniform, and θ the angle described in the time t. Then

$$\theta = \omega t.$$

Obs.—When two straight lines move in one plane the rate at which the angle between them is increasing or decreasing at any instant is called the relative angular velocity of one line with respect to the other. Remarks and formulæ similar to those in the last article apply in this case. It is only necessary to consider angular space instead of linear space.

Def.—The angular velocity of a point with respect to any other point in the plane of its motion is the angular velocity of the line joining the points.

10. *To express the linear velocity of a point moving uniformly in the circumference of a circle in terms of the angular velocity of the point with respect to the centre of the circle.*

Let ω be the angular velocity of the radius joining the moving point to the centre, v the linear velocity of the point moving in the circumference, and a the radius of the circle.

Then the length of the arc described in the unit of time is v units of length, and the corresponding angle described by the radius is ω units of angular measure. Hence if ω be expressed in circular measure

$$\omega = \frac{v}{a}$$

or

$$v = a\omega.$$

Obs.—The time of a complete revolution of the point in the circumference is called the Periodic Time; let its value be T. Then, since 2π is the angle, in circular measure, described uniformly in the time T,

$$\omega = \frac{2\pi}{T}$$

Again, $2\pi a$ is the length of path described uniformly in the time T, hence

$$v = \frac{2\pi a}{T}.$$

11. *If the unit of time be changed in the ratio of* 1 *to* n, *and the unit of space in the ratio of* 1 *to* m, *to find in what ratio the numerical expression for any velocity will be changed.*

Let v be the numerical expression for any velocity, and let it become v' when the units of time and space are changed.

Since v of the old units of space are described in the old unit of time, nv of the old units of space are described in the new unit of time. Hence

$$\frac{nv}{m}$$

of the new units of space are described in the new unit of time. Therefore, by the method of measuring velocity,

$$v' = \frac{nv}{m}$$

Therefore the required ratio is

$$\frac{v}{v'} = \frac{m}{n}.$$

Obs. 1.—The measure of any velocity when the units of time and space are changed may be calculated from the following proportion—

$$\frac{\text{new expression}}{\text{old expression}} = \frac{\text{ratio of new unit of time to old unit}}{\text{ratio of new unit of space to old unit.}}$$

Obs. 2.—The proportion of this article follows immediately from the self-evident proposition that—*the numerical expression for any quantity is inversely proportional to the magnitude of the unit in terms of which the quantity is expressed.*

60 60 1 60 2600 60

For, the fraction space divided by time is changed in the ratio of the unit of time to the unit of space.

EXAMPLES.

1. A point describes uniformly a path 1200 yards in length in an hour; find its velocity in feet per second.

Ans. 1.

2. A point passes over one mile uniformly in one hour; find its velocity in feet per second.

Ans. $\frac{22}{15}$.

3. A railway train travels at the rate of 30 miles per hour; find its velocity in feet per second?

Ans. 44.

4. A railway train travels over 150 miles in 5 hours; what is its average velocity in feet per second.

Ans. 44.

5. The velocity of a point at a given instant is 32 feet per second; find its velocity expressed in miles per hour.

Ans. $21\frac{9}{11}$.

6. Supposing a railway train to travel from Howrah to Delhi, a distance of 955 miles, in 52 hours; what is the average velocity in feet per second?

Ans. 21 8

7. The rate of propagation of sound in air being 1090 feet per second, find after what interval the report of a cannon fired at Calcutta would be heard at Hooghly, the distance being 25 miles?

Ans. 2 min. 1 sec.

8. When a peal of thunder is heard 10 seconds after the flash of lightning which produced it, what is the distance of the thunder cloud, supposing the passage of light to be instantaneous and that sound travels at the rate of 1090 feet per second?

Ans. 2·064 miles.

9. Supposing that light takes eight minutes and twenty seconds to travel from the sun to the earth, a distance of ninety-two millions of miles; calculate the velocity of propagation of light in miles per second.

Ans. 184,000.

10. Supposing the diameter of the earth at the equator to be 7,925 miles, and that it revolves once in 23 h. 56 m.; find the linear velocity of a point at the equator in feet per second.

Ans. 1525·7.

11. Compare the velocities of the extremities of the hands of a clock, the length of the hour hand being two inches, and that of the minute hand 3·5 inches.

Ans. $\frac{1}{21}$.

12. What is the angular velocity of the minute hand of a clock (1) expressed in circular measures per second, (2) in degrees per second?

Ans. $\frac{\pi}{1800}$; $\frac{1}{10}$.

13. Two points move uniformly along the same path; when they move in the same direction their relative velocity is 5 feet per second, and when in opposite directions 25 feet per second; find the velocity of each.

Ans. 15; 10.

14. The wind blowing exactly along a line of railway, two equally quick trains, moving in opposite directions, have the steam track of the one twice as long as that of the other; compare the velocities of the trains and of the wind.

Ans. 3:1.

15. Two railway trains meeting are observed to be 4 seconds in passing each other; had they been moving in the same direction they would have required 12 seconds; supposing the lengths of the trains to be 110 and 130 feet, find at what rate per hour each train is moving.

Ans. $27\frac{3}{11}$; $13\frac{7}{11}$.

16. Two points are at a certain instant at the extremities of the diameter of a circle and are moving uniformly along the circumference with angular velocities ω and ω' circular measures, respectively; find when the points first meet, (1) when they move in opposite directions round the circumference, (2) when they move in the same direction.

Ans. $\frac{\pi}{\omega+\omega'}$; $\frac{\pi}{\omega-\omega'}$.

17. Explain the second result in the last question, (1) when ω' is greater than ω, (2) when they are equal.

18. If a yard be the unit of length and one minute the unit of time, what is the measure of a velocity of three miles per hour?

Ans. 88

19. If the angular velocity of a line be ·7854 circular measures per second, what must be the unit of angular measurement in order that the same velocity may be represented by 90?

Ans. a degree.

20. A man walks with a velocity represented by 2, and he finds that he walks 7 miles in 2 hours; if one foot be the unit of length, what is the unit of time?

Ans. $\frac{30}{77}$ secs.

CHAPTER II.

ACCELERATED VELOCITY.

12. The velocity of a moving point estimated along its path is, as was explained in the last chapter, at any instant measured by the rate of change of the length of path per unit of time, the length of path being reckoned from some fixed point on the path. When the change of length of path in every given time is proportional to the time the velocity is uniform in magnitude, and when this is not the case the velocity is variable.

All the circumstances of the motion of a point, along a *given* path, can in general be determined when the position and velocity of the point at any given instant are known, and when either the law of change of velocity or the law of change of length of path is given. We shall now consider the simplest case of a variable velocity, namely when the change of velocity is uniform or constant.

13. UNIFORM ACCELERATION.—When the velocity of a point increases it is said to be accelerated; when it decreases, retarded. If the increase or decrease of the velocity in every given time be proportional to the time the velocity is said to be uniformly accelerated or retarded, and the amount of increase or decrease of the velocity in the unit of time is called the *acceleration* or retardation. It is convenient to consider a retardation as a negative acceleration, and the word *acceleration* will be used in both senses, its quantity being considered positive or negative.

Acceleration, when uniform, is measured by the actual increase of the velocity in the unit of time, and the unit of acceleration is the acceleration when a unit of velocity is added (or subtracted) in the unit of time.

Let a be the measure of a uniform acceleration, and v the change of velocity in the time t, then, since a is the change of the velocity in the unit of time,

$$v = a\,t.$$

When the velocity of a point changes uniformly the acceleration is constant; that is, it is the same in amount at every instant of the motion, and its numerical value may be derived from the formula

$$a = \frac{v}{t}$$

where v is the velocity actually added in *any* given time t. Thus we shall obtain the same result whether we derive a from the velocity added in a million seconds, or from the velocity added in the one-millionth part of a second. The fraction, change of velocity divided by the time of change, expresses the velocity which would be added in a unit of time at the same rate, no matter how small the time be taken.

14. VARIABLE ACCELERATION.—When the change of velocity is not proportional to the time of change; that is, when the velocity does not change uniformly, the rate of change is different at different successive instants of time. It is clear, however, that at any given instant the velocity is changing at a definite rate, and this rate of change of velocity is the measure of the acceleration at that instant.

Acceleration, when variable, is measured at any instant by the velocity which would be added in a unit of time supposing the rate of change of velocity to continue uniform and the same as at the instant under consideration.

This statement is in fact nothing more than the equation for uniform acceleration

$$a = \frac{v}{t}$$

expresses when the acceleration is derived from the velocity added in an indefinitely short time, during which time the rate of change of the velocity may be considered uniform.

The acceleration may then always be derived from the above equation, but when the acceleration is variable we must derive it, at any instant, from the velocity added in an indefinitely short time reckoned from the instant.

Obs. 1.—The student will have observed the similarity of the methods of measuring velocity and acceleration. The one is measured by the rate of change of space per unit of time and the other by the rate of change of velocity per unit of time.

Obs. 2.—Acceleration of angular velocity and acceleration of relative velocity (uniform and variable) are measured by methods exactly analogous to those which have been explained, and it is unnecessary to repeat definitions and statements which only require obvious alterations in order to make them applicable to the above cases.

Def.—The average acceleration during any time is the change of velocity in that time divided by the time.

15. *The velocity of a point, estimated along its path, is uniformly accelerated; given the point's velocity at any instant; to find its velocity at any other instant.*

Let V be the velocity of the point at the given instant, a the measure of the acceleration, and v the velocity at the end of a time t, reckoned from the instant.

Then $a\,t$ units of velocity are added in the time t, Art. 13, hence

$$v = V + a\,t.$$

Obs.—When the point's velocity is retarded, a is to be taken as negative. Again, if we seek the velocity of the point t units of time before the instant when the velocity is V, t must be taken negative.

16. *A point's velocity, estimated along its path, is uniformly accelerated by an amount* a *per unit of time; given the velocity* v *at any instant to find the space* s *passed over in the time* t.

Since the velocity increases uniformly, the velocity at any time *before* the middle of the interval t must be as much *less* than the velocity at the middle of the interval as the velocity at the same time *after* the middle of the interval is *greater* than the velocity at the middle of the interval. Hence the velocity at the middle of the interval t is the average velocity during that interval, and is also equal to the arithmetic mean of the initial and final velocities. Thus, Art. 15,

$$\text{average velocity} = \tfrac{1}{2}\,(V + V + a\,t),$$

but $$\text{average velocity} = \frac{s}{t};$$

therefore $$\frac{s}{t} = V + \tfrac{1}{2}a\,t,$$

or $$s = Vt + \tfrac{1}{2}a\,t^2,$$

which equation determines the space required.

Obs.—When the point's velocity is retarded, a is to be taken negative.

17. The following is another proof of the important equation obtained in the last article.

Suppose the whole time t to be divided into n equal parts. Then, since in each unit of time the velocity is increased by a, the velocities of the point at the end of the times

$$\frac{t}{n},\ 2\frac{t}{n},\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \ n\frac{t}{n},$$

from starting are, respectively,

$$V + a\frac{t}{n},\ V + 2a\frac{t}{n},\ \ldots\ldots\ldots\ldots\ldots\ \ V + na\frac{t}{n}.$$

Let s_1 be the space which the point would describe if it moved during each of the equal intervals uniformly with the velocity which it has at the beginning of the interval; and let s_2 be the space which the point would describe if it moved during each of the equal intervals with the velocity which it has at the end of the interval. Then,

$$s_1 = V\frac{t}{n} + \left(V + a\frac{t}{n}\right)\frac{t}{n} + \left(V + 2a\frac{t}{n}\right)\frac{t}{n} + \ldots\ldots\ldots$$
$$+ \left(V + (n-1)a\frac{t}{n}\right)\frac{t}{n};$$

$$s_2 = \left(Vt + a\frac{t}{n}\right)\frac{t}{n} + \left(V + 2a\frac{t}{n}\right)\frac{t}{n} + \ldots\ldots\ldots + \left(V + na\frac{t}{n}\right)\frac{t}{n};$$

therefore $$s_1 = Vt + a\frac{t^2}{n^2}\,(1 + 2 + 3 + \ldots\ldots\ldots + n - 1)$$

$$= Vt + a\frac{t^2}{n^2}\,\frac{n\,(n-1)}{2} = Vt + \frac{at^2}{2}\left(1 - \frac{1}{n}\right)$$

$$s_2 = Vt + a\frac{t^2}{n^2}(1+2+3+\ldots\ldots\ldots+n)$$

$$= Vt + a\frac{t^2}{n^2}\,\frac{n(n+1)}{2} = Vt + \frac{at^2}{2}\left(1+\frac{1}{n}\right)$$

The space s, actually described by the point in the whole time t, must evidently lie between s_1 and s_2; but by making n large enough $\frac{1}{n}$ can be made less than any assigned quantity; so that s_1 and s_2 become more nearly equal to $Vt+\frac{1}{2}at^2$ the greater the value of n. Hence, since s must always lie between s_1 and s_2, we have

$$s = Vt + \tfrac{1}{2}at^2.$$

18. The equations of Articles 15 and 16,

$$v = V + at \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$s = Vt + \tfrac{1}{2}a\,t^2 \ldots\ldots\ldots\ldots\ldots \quad (2)$$

are sufficient for the solution of all questions concerning the motion of a point whose velocity, estimated along a *given* path, is uniformly accelerated.

Eliminating t between equations (1) and (2) (which may be done by squaring (1), multiplying (2) by $2a$ and subtracting) another important equation

$$v^2 = V^2 + 2as \ldots\ldots\ldots\ldots\ldots \quad (3)$$

is obtained, which determines the velocity in terms of the initial velocity and space described.

Obs.—The equations (1), (2) and (3), and the exact meanings of the symbols, should be carefully remembered. V is, in every case, the velocity of the point at the beginning of the time t, or is, in other words, the initial velocity; v is the velocity at the end of the time t, or, it is the velocity after the point has passed over a space s; a is the acceleration, and may be positive or negative. When the velocity of the point is retarded, the negative sign must be used for a in the above equations.

19. When the point starts from rest at the commencement of the time t, that is when $V=0$, the equations of the last article become

$$v=a\,t \qquad (1)$$

$$s=\tfrac{1}{2}a\,t^2 \qquad (2)$$

$$v^2=2as \qquad (3)$$

Since a is a constant quantity, the laws of the motion of a point starting from rest, and whose velocity is uniformly accelerated along the path of motion, may be expressed in words, thus:—

(1). The velocity acquired in any time is proportional to the time.

(2). The spaces passed over are proportional to the squares of the times elapsed since the commencement of the motion.

(3). The spaces passed over from the commencement of the motion are proportional to the squares of the velocities acquired.

Conversely, if a point start from rest, and if we observe that any of the above conditions are fulfilled, we may conclude that the point's velocity is uniformly accelerated.

Obs.—The velocity acquired by a point, which starts from rest, in passing over a space s, is $\sqrt{2as}$, and is called the velocity due to the space s.

20. The equation of Art. 16, and the results derived therefrom, might also be investigated in the following manner.

A point starting from rest moves in such a manner that the spaces described are proportional to the squares of the times elapsed since the commencement of the motion; to show that the point's velocity is uniformly accelerated, and to find the space described in any time,

Let s_1, s and s_2 be the spaces described in the times $t-h$, t and $t+h$, where h is indefinitely small. Then, by hypothesis,

$$s_1=k\,(t-h)^2\,;\;\; s=kt^2\,;\;\; s_2=k\,(t+h)^2\,;$$

where k is a constant quantity whose value is to be determined. From these equations we get

$$s_2 - s_1 = 4kht.$$

Therefore $$\frac{s_2 - s_1}{2h} = 2kt.$$

Now $s_2 - s_1$ is the space described in the time $2h$: let v be the velocity of the point at the end of the time t; then when h is indefinitely small, we have, Art. 6,

$$v = \frac{s_2 - s_1}{2h};$$

therefore $$v = 2kt.$$

This equation proves that the velocity is uniformly accelerated, since the velocity acquired in any time is proportional to the time (Art. 13). Also, if a be the measure of the acceleration; that is, the velocity added in the unit of time, we have

$$a = 2k,$$

and $$s = kt^2;$$

therefore $$s = \tfrac{1}{2}at^2,$$

which is the second equation of Art. 19.

Again, when the point does not start from rest, but has a velocity V at the commencement of the time t, let τ be the time from rest up to the commencement of the time t, so that $V = a\tau$. Then,

space described from rest in time $\tau + t = \frac{1}{2}a(\tau + t)^2$;

" " " " $\tau = \frac{1}{2}a\tau^2$;

space described in time $t = s = \frac{1}{2}a(2\tau t + t^2)$.

Substituting for $a\tau$ its value V, we have

$$s = Vt + \tfrac{1}{2}at^2,$$

which is the equation of Art. 16.

Cor.—We have $s = kt^2 = \frac{1}{2}vt$.

Hence the space described is half the product of the time and final velocity.

21. *To find the space described by a point, whose velocity is uniformly accelerated, in the* nth *unit of time.*

We have, by the equation of Art. 16,

$$\text{space in } n \text{ units} = n\ V + \tfrac{1}{2}\,a\,n^2\,;$$

$$\text{space in } n-1 \text{ units} = (n-1)\ V + \tfrac{1}{2}\,a\,(n-1)^2\,;$$

therefore, by subtraction,

$$\text{space in the } n\text{th unit} = V + \tfrac{1}{2}\,a\,(n^2 - (n-1)^2)$$

$$= V + \tfrac{1}{2}\,a\,(2n-1).$$

When the point starts from rest at the commencement of the time n

$$\text{space in the } n\text{th unit} = \tfrac{1}{2}\,a\,(2\,n-1).$$

Cor.—The spaces described in successive units of time by a point which starts from rest, and whose velocity is uniformly accelerated, are proportional to the series of odd numbers 1, 3, 5......$2n-1$.

22. *To show that the numerical expression for an acceleration varies directly as the square of the unit of time, and inversely as the unit of space.*

Let a be the numerical value of the acceleration when the units of time and space are one second and one foot respectively, and let the new units be n seconds and m feet respectively.

The acceleration a is a velocity of a feet per second added in one second. But a velocity of a feet per second is the same as a velocity $\frac{na}{m}$ (m feet) per n seconds, and this velocity is added in one second. Therefore in n seconds a velocity of $\frac{n^2a}{m}$ (m feet) per n seconds is added.

Hence $\frac{n^2a}{m}$ units of velocity are added in the unit of time; this is, therefore, the numerical value a' of the same acceleration when the units are changed. Thus

$$\frac{a'}{a} = \frac{n^2}{m}$$

Obs. 1.—We have taken, for the sake of clearness, the original units of time and space to be a second and a foot, respectively. This however is, of course, not essential; and in general when the unit of time is changed in the ratio of 1 to n and the unit of space in the ratio of 1 to m, the new measure for the acceleration may be calculated from the proportion:—

$$\frac{\text{new expression}}{\text{old expression}} = \frac{\text{ratio squared of new unit of time to old unit}}{\text{ratio of new unit of space to old unit}}$$

Obs. 2.—The proposition of this article also follows immediately from equation (2) of Art. 18, written thus

$$a = 2\frac{s}{t^2}$$

For, as explained in Art. 11, *Obs.* 2, the fraction space divided by square of time is changed in the ratio of the square of the unit of time to the unit of space. See, also, the next article.

23. The following is another method of arriving at the results of the last article.

Let a be the measure of an acceleration when the units of time and space are, respectively, n seconds and m feet; and let a' be the measure of the same acceleration when the units are n' seconds and m' feet. Also let σ feet be the space over which the acceleration of velocity would carry the point from rest in τ seconds.

The numbers which express the space s and the time t in terms of the first units are, respectively, $\frac{\sigma}{m}$ and $\frac{\tau}{n}$. Hence, since $s = \frac{1}{2}\, a\, t^2$,

$$\frac{\sigma}{m} = \frac{1}{2}\, a \left(\frac{\tau}{n}\right)^2.$$

Similarly $$\frac{\sigma'}{m'} = \frac{1}{2}\, a' \left(\frac{\tau'}{n'}\right)^2.$$

Therefore $$\frac{a'}{a} = \left(\frac{n'}{n}\right)^2 \div \frac{m'}{m}.$$

We shall now give a few examples of the application of the formulæ of this chapter.

24. Ex. 1.—*A point has its velocity, along its path, uniformly retarded; to find when and where it will come to rest.*

We must give a the negative sign in the equation of Art. 15, and when the point's velocity is zero, we get

$$v = 0 = V - at.$$

Hence $$t = \frac{V}{a}$$

gives the time when the point comes to rest.

Again $$v^2 = 0 = V^2 - 2\,a\,s.$$

Hence $$s = \frac{V^2}{2a}$$

determines the place where the point comes to rest, s being measured from the position of the point when $t = 0$.

Cor.—When the point comes to rest, the retardation of velocity becomes an acceleration from rest in the opposite direction, and the point will return to its original position in a time found from the equation

$$s = \frac{V^2}{2a} = \tfrac{1}{2}\,a\,t^2;$$

from which $$t = \frac{V}{a}.$$

Hence, the times of moving to the greatest distance from a given point and then returning to the original position are equal. In other words, the whole time which elapses until the return of the point to its original position is twice the time of arriving at the greatest distance.

Obs.—When the point's velocity is uniformly accelerated a is positive and the supposition $v = 0$ gives

$$t = -\frac{V}{a};\ s = -\frac{V^2}{2a}.$$

The first result gives the time, *before* the instant from which the time is reckoned, when the point was at rest; the second gives the space passed over from rest up to the position of the point at the instant from which we reckon.

25. Ex. 2.—*A point starts with a given velocity* V *from a point* O, *and its velocity is uniformly retarded along its path; to find the time when the point passes through a given point* A *on its path, and to discuss the different cases which may arise.*

Let the distance, measured along the path, from O to A be a; then making $s = a$ in the general equation

$$s = Vt - \tfrac{1}{2}at^2$$

we get, to determine the time of arriving at A, the quadratic

$$at^2 - 2Vt + 2a = 0.$$

As this equation gives in general two values for t, the point will twice pass through A, if the roots are real and positive.

The roots of the above equation will be real when $V^2 > 2aa$, and they must then be both positive, since the coefficient of the second term is negative and the coefficients of the other terms positive.

When, however, $V^2 = 2aa$, the values of t are equal, and the moving point will just arrive at A.

When $V^2 < 2aa$ the roots of the equation are impossible, and the point will, hence, never arrive at A.

In the first case the values of t are

$$\frac{V \mp \sqrt{V^2 - 2aa}}{a}$$

The smaller value gives the time when the point first arrives at A, and the greater value gives the time when the point again passes through A, when it is returning to the origin O.

In the second case the value of t is $\frac{V}{a}$, and this is the time of arriving at the greatest distance. (See also Art. 24.) The point will return to the origin O in the time $\frac{2V}{a}$ which may be got by making $s = 0$ in the general equation. (See also Art. 24 *Cor.*)

Obs. 1.—We may also deduce in a different manner the values of t when the point passes through A.

Let B be the point on the path at which the moving point comes to rest, and let $OB = h$. Then, as in Art. 24,

$$\text{time through } OB = \frac{V}{a}; \quad h = \frac{V^2}{2a}.$$

When $a < h$; that is, when $V^2 > 2aa$ the moving point, having passed through A, will come to rest at B; after this the retardation becomes an acceleration from rest towards the origin O.

The time from A to B is, by Art. 24 *Cor.*, equal to the time from B to A, and may be found by substituting for h its value (given above) in th equation

$$h - a = \tfrac{1}{2}at^2.$$

Adding and substracting the value of t from this equation to the time through OB, we get the same results as before.

Obs. 2.—In the following examples, whenever the units of space and time are not expressed, they are to be taken as one foot and one second, respectively.

EXAMPLES.

1. A point's velocity is uniformly accelerated by an amount 5; find the time of acquiring a velocity 20, (1) when the point starts from rest, (2) when the initial velocity is 5.

Ans. 4 secs; 3 secs.

2. A point starts with a velocity 12 and acquires a velocity 52 in 4 seconds, find the acceleration supposing it to be uniform.

Ans. 10.

3. A point has, at a given instant, a velocity 96, and its velocity is retarded by an amount 32; find when it will come to rest.

Ans. 3 secs.

4. Find the space described, by a point whose velocity is accelerated by an amount 32·2, in 3 seconds, (1) when the point starts from rest, (2) when the initial velocity is 10.

Ans. 144·9; 174·9.

5. A point passes over 18 feet in 2 seconds, the initial velocity being 15; find the acceleration, supposing it to be uniform.

Ans. −6.

6. If the initial velocity be 15 and the acceleration -6; find the time of describing 6 yards. (See Art. 25.)

Ans. 2 secs.

7. A point starts with a velocity 10, and having passed over 3 feet its velocity is 8; find the acceleration.

Ans. -6.

8. A point, whose velocity is retarded by an amount 32, has at starting a velocity 80; find the space it will pass over before it comes to rest.

Ans. 100 ft.

9. If the acceleration be 300 yards per minute, over what space must a point, starting from rest, pass so as to acquire a velocity of 30 feet per second?

Ans. 1,800 ft.

10. A point whose velocity is uniformly accelerated by an amount a has initially a velocity V; if v be the velocity and s the space described at the end of the time t, prove that

$$\frac{v-V}{a} = \frac{2s}{v+V} = t.$$

11. If a velocity of 30 miles an hour be the unit of velocity, and a mile be the unit of space; find the number which represents an acceleration 32 feet per second.

Ans. 87·27.

12. An acceleration is represented by 32·2 when the units of time and space are one second and one foot; find what number will represent the same acceleration when the units of time and space are one minute and one mile.

Ans. 21·95.

13. If one foot be the unit of length, and the acceleration 32·2 feet per second the unit of acceleration, what is the unit of time?

Ans. 0·17 secs.

14. A point describes 75 feet from rest and, with a uniform acceleration 8, acquires a velocity 20 in 5 seconds; what are the units of time and space?

Ans. 2 secs; 3 feet.

15. A point has its velocity uniformly accelerated by an amount a; if u be the arithmetic mean of the first and last velocities, in passing over any portion h of the path, and if u' be the velocity gained, prove that $uu' = ah$.

16. A point moving in a line has velocities v_1, v_2, v_3............at the end of equal intervals of time; in what case is the acceleration uniform, and what then is its measure?

Ans. v_1, v_2, v_3............must be in A. P., the acceleration being the ratio of the common difference to the length of interval.

17. A point moves over 7 feet in the first second of time during which it is observed, and over 11 and 17 feet in the third and sixth seconds, respectively: prove that the acceleration is constant, and find its value.

Ans. 2.

18. A point describes in successive intervals of 4 seconds the spaces 24 and 64 feet; determine the acceleration, and the velocity at the commencement of the first interval.

Ans. $V = 1$; $a = 2{\cdot}5$.

19. A point moves round a closed curve with uniformly accelerated velocity; in the mth nth and pth seconds it describes x, y, z circuits, respectively, prove that

$$x(n-p) + y(p-m) + z(m-n) = 0$$

20. If a point, whose velocity is uniformly accelerated in a right line, start from rest and describe a feet in the mth second reckoned from the commencement of the motion; find the space described in the xth second.

Ans. $\dfrac{2x-1}{2m-1}\,a$.

21. If a point, whose velocity is uniformly accelerated in a right line, describe a feet and b feet in the mth and nth seconds, respectively; find the space described in the xth second, the initial velocity being V.

Ans. $V + \frac{1}{2}\,\dfrac{a-b}{m-n}\,(2x-1)$.

22. A point, whose velocity is uniformly accelerated in a right line, starts from rest and describes a feet in the mth second and b feet in the nth second; show that the space described in the $(m+n)$th second is

$$a + b + \frac{a-b}{2(m-n)}.$$

23. A point A starts from a given point and moves in a right line with uniform acceleration a. If n seconds after A has started a second point B starts from the same point, moving with uniform velocity V, and overtakes A, show that V cannot be less than $2an$, and prove that if V be greater than this the points will pass each other twice.

CHAPTER III.

KINEMATICAL PRINCIPLES AND METHODS.

26. In the last two chapters we considered the amount of velocity of a moving point, and the rate of change of the amount of velocity (acceleration of velocity estimated along the path of motion), but nothing was said respecting the direction of motion at any instant, which changes continually when the point does not move in a straight line.

When a point moves along any path, the direction of its velocity at any instant is the direction in which the point is moving at that instant, and is the tangent to the path drawn at the position which the point occupies at the instant. Hence, when a point moves in a straight line, and not otherwise, the direction of its velocity is uniform, or constant; and the velocity of a point is to be considered uniform, only, when it is uniform in magnitude and constant in direction; that is, when the point describes a right line uniformly.

27. **Geometrical Representations.**—When the velocity of a point is uniform (in magnitude and direction), it may be represented at any instant, geometrically, by the portion of the straight line which would actually be described by the point in the next unit of time. This portion contains as many units of length as the amount of the velocity contains units of velocity (Art. 2).

When the velocity of a point is variable (in magnitude, or in direction, or in both), it may be represented at any instant, geometrically, by the portion of the straight line which would be described by the point in the next unit of time, supposing the velocity of the point were to continue uniform (in magnitude and direction) during the unit of time and the same as at the instant under consideration.

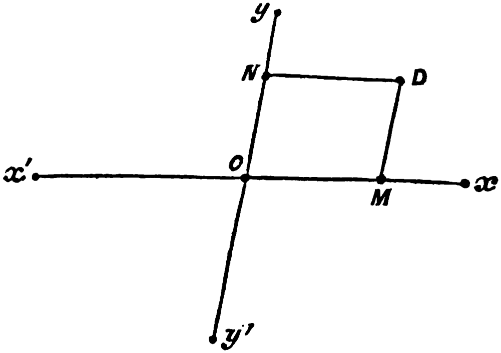
When a point moves in a *straight line*, the acceleration of its velocity at any instant is measured by the velocity which would be added in the next unit of time, supposing the acceleration were to continue uniform for that unit of time, and the same as at the instant under consideration. Hence, the acceleration may be represented at any instant by that portion of the line of motion, measured from the point's position which represents the velocity which would be added (at the same rate) in the next unit of time. This line will contain as many units of length as the acceleration at the instant contains units of acceleration.

When a point moves in a curve, the acceleration of the magnitude of its velocity may be similarly represented at any instant by a portion of the line of motion, at that instant, measured from the point and containing as many units of length as the acceleration of the amount of velocity contains units of acceleration. It will be explained, however, further on, that when a point moves in a curve, the acceleration of the amount of velocity is not the whole acceleration of velocity.

Obs. 1.—The student should be careful not to fall into the error of supposing the portion of a line which represents an acceleration to be the space which *would* be described by the point in the unit of time. In no case is this true; in fact, if the point were to start from rest and move in a right line with uniformly accelerated velocity, the space described in the first unit of time would, by equation (2) of Art. 19, be half the length of the line which would represent the acceleration at the commencement of the motion.

Obs. 2.—The velocity of a point may be represented in amount and direction, but not in situation, by a line drawn through *any* given point parallel to the line of motion, and containing as many units of length as the velocity contains units of velocity. Again, instead of taking the unit of length to represent the unit of velocity, we may take any convenient multiple or sub-multiple of the unit of length to represent the unit of velocity. Similar remarks apply to acceleration of velocity.

Obs. 3.—As there are two opposite directions of motion along a line, an arrow-head on the line which represents a velocity, or acceleration, may be conveniently used to indicate which direction is intended to be represented.



28. COMPONENT VELOCITIES.—Let D be a point moving in any manner along any path in one plane, and let xx' and yy' be any two fixed lines in the plane intersecting at O.

From D let lines DN and DM be drawn parallel to Ox and Oy, respectively, and as D moves, let these lines be supposed to move so as always to continue parallel to Ox and Oy. Then as D moves, M moves along Ox, and N along Oy, and, instead of studying the motion of D directly, it is often more convenient to study the motions of M and N, and from their motions and positions at any time, to infer the actual motion of D and its position. See also *Statics*, Art. 80.

The rate at which M is moving along Ox at any instant; that is, the rate at which the co-ordinate OM is changing, is called the component of the velocity of the point parallel to Ox; so the rate at which N is moving at the instant is called the component velocity parallel to Oy. The component velocities (when uniform and when variable) are to be measured in the usual manner.

The rates at which the component velocities are changing at any instant are called the components of the acceleration of the point's velocity. These component accelerations of velocity (when uniform and when variable) are to be measured in the usual manner.

Obs. 1.—We may otherwise consider the circumstances of the motion of the moving point to be determined by the motion of an indefinite line DM, which moves parallel to Oy, and the motion of a point D which moves along the moving line MD.

Obs. 2.—When a point moves in any manner, not necessarily in one plane, its position is to be referred to three planes which determine by their intersections three lines meeting in a point, *Statics*, Art. 80. Thus in the diagram, *Statics*, Art. 61, the rates at which the co-ordinates x, y, z of D change are the components of the velocity of D parallel to Ox, Oy and Oz, and the rates at which the componet velocities change are the components of the acceleration of velocity.

29. RECTANGULAR RESOLUTIONS.—When the angle between the given directions is a right angle, the component velocities may be defined *independently* of each other, and the same is true for the component accelerations of velocity. In this case the angle xOy is a right angle (Fig. Art. 28), and DM and DN are perpendiculars on Ox and Oy, respectively. Thus—

The *resolved* velocity of a point parallel to a given direction is the velocity of the foot of the perpendicular drawn from the point on the given direction.

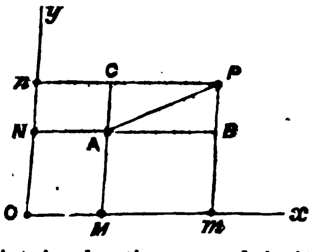
The *resolved* acceleration of a point's velocity parallel to a given direction is the acceleration of the velocity of the foot of the perpendicular drawn from the point on the given direction.

Obs. 1.—The terms *resolved velocity* and *resolved acceleration* will be used only in the sense above explained.

Obs. 2.—When the co-ordinates of a point are given at any instant, the position of the point is known. It is now to be explained how, from the direction and magnitude of the velocity of a point at any instant, the component velocities are to be inferred, and, conversely, how the magnitude and direction of a point's velocity can be determined at any instant when the point's component velocities are known.

30. *When a point's velocity is uniform (in magnitude and direction) its component velocities parallel to any two fixed directions are uniform.*

In this case the point moves along a straight line uniformly. Let AP be a portion of this straight line, A being the position of the point at any instant, and P its position at the end of a time t reckoned from that instant.



Through A and P draw parallels to the fixed lines Ox and Oy. Then $AB = Mm$ and $AC = Nn$ are the increments of the co-ordinates of the point in the time t, and it is required to show that these increments are proportional to the time t.

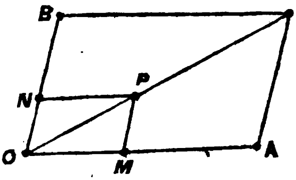
Now as the point P moves along the line AP while the line PBm moves so as always to be parallel to Oy, the triangle ABP remains equiangular, and therefore similar to itself. Hence, the ratio of AB to AP is constant; but, if v be the constant velocity of P, we have $AP = vt$. Therefore AB, or Mm, is proportional to vt; that is, to t. Similarly it may be shown that AC, or Nn, is proportional to t. Hence, the component velocities of P are uniform.

Cor. 1.—If we take the time $t = 1$, AP will represent the velocity of the point in magnitude and direction (Art. 27), and the sides AB and AC of the parallelogram $ABPC$ will represent the component velocities.

Cor. 2.—When a point's velocity is variable it is represented at any instant by the line AP, drawn from its position, which would be described, supposing the velocity were to remain unchanged in magnitude and direction for the unit of time. Hence, the sides AB and AC of the parallelogram $ABPC$ will represent the component velocities of the point at the instant under consideration.

Obs.—It might be shown, as in this article, that, conversely, when the component velocities of a point are uniform, the point's actual velocity is uniform; that is, that the point describes a right line uniformly. This, however, will be shown in the next article under a somewhat different statement.

31. Parallelogram of Velocities.—*If the components of the velocity of a point parallel to two given directions be represented by straight lines drawn from the position of the point at any instant, the actual velocity of the point will be represented in magnitude and direction by that diagonal of the parallelogram constructed upon these lines, which passes through the point.*



First, suppose the component velocities to be uniform.

Let O be the position of the point at the instant under consideration. Draw OA and OB parallel to the given directions and representing the component velocities of the point. Complete the parallelogram $OACB$.

Let P be the position of the point at any other instant. Draw PN and PM parallel to the given directions, and therefore parallel to OA and OB, respectively. Then, as P moves the rates at which OM and ON increase are equal to the component velocities of P parallel to the given directions. Hence, by hypothesis, OM and ON increase uniformly with the time, and at the end of a unit of time are equal to OA and OB, respectively.

Hence during the motion, always,

$$\frac{OM}{ON} = \frac{OA}{OB}.$$

Therefore, by *Euclid* VI, 26, the parallelograms $OMPN$ and $OACB$ are about the same diagonal OC.

Again, as P moves, OP has to OM the constant ratio of OC to OA; but OM increases in proportion to the time; therefore OP increases in proportion to the time, and at the end of the unit of time is equal to OC.

Therefore P describes the line OC uniformly in the unit of time; hence, OC represents in magnitude and direction the actual velocity of P.

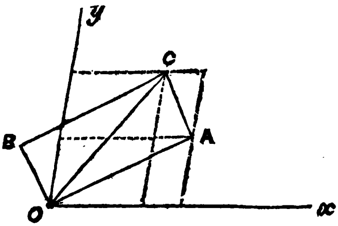
When the component velocities are variable, OA represents the space which would be described by M in the unit of time if the velocity of M were to remain unchanged for that unit of time, and the same as at the instant when M coincided with O. So OB represents the space which would be described by N; and hence, as before, OC represents the space which would be described by P in the unit of time if its velocity were to remain unchanged, and the same as when P coincided with O; which proves the proposition.

Obs. 1.—We have supposed the lines which represent the components of the point's velocity to be drawn from the position of the point. They may, however, be drawn from *any* given point, and then the diagonal of the parallelogram will represent the actual velocity of the point in magnitude and direction, but not in situation. Again, it is not necessary to take OA and OB equal to the increments (or what would be the increments) of the co-ordinates of P in the unit of time. It will be sufficient if OA and OB be taken *proportional* to those increments, and then OC will be *proportional* to the actual velocity; that is, will represent it on the same scale that OA and OB represent the component velocities.

Obs. 2.—The actual velocity of a point is called the *resultant* velocity, with reference to its component velocities.

Obs. 3.—Similarly it may be shown, when the velocity of a point is referred to three directions meeting at a point, but not lying in the same plane, that the diagonal of the parallelopiped constructed on the lines representing the component velocities represents the resultant velocity in magnitude and direction. See also *Statics*, Art. 61.

32. CHANGE OF VELOCITY.—From any point O, let OA be drawn, representing the velocity of a moving point in magnitude and direction (not in situation), and let OC be drawn to represent the velocity of the same point after the end of an interval of time t. Join AC, and complete the parallelogram $OACB$.



Then the velocity of the point at the end of the interval is equivalent to component velocities represented by *OA* and *OB*. But if the velocity of the point had not changed in the interval, it would have been represented by *OA* at the end of the interval. Hence *OB*, or *AC*, represents in magnitude and direction the change of velocity in the interval *t*.

Instead of representing the velocities at the commencement and end of the interval by *OA* and *OC*, we may represent them by their component velocities parallel to *Ox* and *Oy*, and it immediately appears from the diagram that *AC* is the diagonal of the parallelogram constructed on lines drawn through *A* representing the changes of the component velocities. So *OB* is the diagonal of the parallelogram constructed on lines drawn through *O* representing the changes of the component velocities.

When a point moves in a right line, the change of velocity is in the direction of motion, and is simply the increase or decrease of the amount of velocity in the given interval of time. This simple case is also included in the construction given above, *OC* being then measured along the same line as *OA*, and *AC* will represent the change of velocity.

Obs.—Change of velocity, whether arising from change in the amount of velocity, or from change in the direction of velocity, or from both, is to be understood in the sense here explained.

33. Accelerated Velocity.—It follows from Arts. 30 and 31, that the component velocities of a point parallel to any two fixed directions are uniform when the point's velocity is uniform in magnitude and direction (that is when the point describes a straight line uniformly), and not otherwise. It also follows from the construction of the last article that the velocity of a point changes except in the case when the point moves uniformly in a right line.

Acceleration of velocity at any instant is defined in general as the rate of change of velocity at that instant, whether arising from a change in the amount of velocity, or from a change in the direction of velocity, or from both; and the direction of acceleration at any instant is the direction in which the velocity is changing at that i

Hence, the acceleration of velocity at any instant may be represented in magnitude and direction by the line which represents the change of velocity which would take place in the next unit of time, if the rate of change were to continue uniform in amount and constant in direction for that unit of time, and the same as at the instant under consideration.

34. PARALLELOGRAM OF ACCELERATIONS.—*If the accelerations of the component velocities of a point parallel to any two given directions be represented at any instant by straight lines drawn from any point, the actual, or resultant, acceleration of the point's velocity will be represented by the diagonal of the parallelogram constructed on these lines which passes through the point.*

For, the component accelerations of velocity are measured by what would be the changes of the component velocities in the next unit of time, supposing that the rates of change were to continue uniform for that unit of time and the same as at the instant under consideration.

Hence, the diagonal of the parallelogram represents in magnitude and direction (not in situation) what would be the resultant change of velocity in the next unit of time, if the change were to continue uniform in rate and constant in direction for that unit of time; that is, it represents the resultant acceleration.

Obs.—The resultant acceleration of velocity is thus to be found from the component accelerations in the same manner that the resultant velocity is found from its components, and the formulæ which we shall now give for determining the resultant velocity from its components and *vice versâ* may be made applicable to the case of accelerations by merely changing the symbols which stand for velocities into symbols for accelerations.

35. *Given the component velocities* v_1 *and* v_2 *of a point at any instant, parallel to any two given directions inclined at an angle* θ*; required to determine the resultant velocity* v *and the angles which it makes with the given directions.*

Draw OA and OB (Fig. Art. 31) parallel to the given directions and representing the component velocities v_1

and v_2, respectively, at any instant. Complete the parallelogram *OACB*; then *OC* represents v in magnitude and direction.

We have, angle $AOB = \theta$; let angle $AOC = \alpha$; angle $BOC = \beta$. Then

$$\frac{OA}{v_1} = \frac{AC}{v_2} = \frac{OC}{v}.$$

Therefore $$\frac{v_1}{\sin\beta} = \frac{v_2}{\sin\alpha} = \frac{v}{\sin\theta}.$$

Also $$OC^2 = OA^2 + AC^2 + 2OA\cdot AC\cos\theta.$$

Therefore $$v^2 = v_1{}^2 + v_2{}^2 + 2v_1 v_2 \cos\theta.$$

These equations are sufficient to determine the resultant velocity and the angle which it makes with a given direction at any instant.

Obs.—The sides of the triangle *OAC* (or *OBC*) represent the resultant velocity and its components; two of the angles are equal to the angles which the direction of the velocity makes with the given directions, and the third angle is the angle between the given directions. Hence, when any three of the six quantities (except the three angles) are given, the remaining quantities can be found by the rules of Trigonometry.

36. Resolved Components.—The case when the angle between the given directions is a right angle, deserves special notice (Art. 29). We may deduce the formula in this case from those in the last article by making $\theta = 90°$ and $\sin\beta = \cos\alpha$. Or, they may be deduced directly from the diagram, Art. 31, when the angle *AOB* is taken as a right angle. Thus

$$v_1 = v\cos\alpha;\ v_2 = v\sin\alpha.$$

Therefore $$\tan\alpha = \frac{v_2}{v_1};$$

and $$v^2 = v_1{}^2 + v_2{}^2.$$

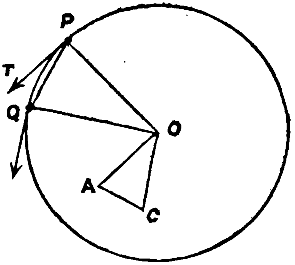
These equations determine the resultant velocity and the angle which its direction makes with a given direction at any instant.

Cor. 1.—The resolved velocity (or acceleration of velocity) of a point parallel to a given direction is found by multiplying the velocity (or acceleration of velocity) by the cosine of the angle which its direction makes with the given direction.

Cor. 2.—When the velocity (or acceleration of velocity) is represented by a line, the resolved velocity (or acceleration of velocity) is represented by the *projection* of the line on the given direction.

Cor. 3.—Hence, it may be immediately shown geometrically, by projecting the sides and diagonal of a parallelogram on any given direction, that the resolved part of a velocity (or of an acceleration of velocity) in any direction, is equal to the sum of the resolved parts of its components in the same direction. This is otherwise evident, since the velocity (or acceleration of velocity) of a point is completely represented by its components.

37. *A point moves in a circle with uniform velocity; to determine the amount and direction of its acceleration of velocity at any instant.*



Let P be the position of the point in the circle at any instant, and let Q be its position at the end of a time t.

From the centre O of the circle draw OA and OC representing in magnitude and direction the point's velocity when at P and at Q, respectively. Then OA is parallel to the tangent at P, and OC is parallel to the tangent at Q; also, since the velocity is constant in magnitude, OA is equal to OC.

Now, by Art. 32, AC represents the change of the point's velocity, in magnitude and direction, in the time t, and, as the triangles AOC and POQ are evidently similar, we have

$$\frac{AC}{AO}=\frac{PQ}{PO}$$

Therefore $$\frac{AC}{t}=\frac{AO}{PO}\,\frac{PQ}{t}.$$

But when the time t is taken indefinitely small,—that is, when Q is indefinitely near to P,—the chord PQ becomes an indefinitely small arc of the circle, and AC represents the change of velocity in an indefinitely small time t. Hence, if v be the amount of velocity in the circle, and a the amount of acceleration of velocity at P, we have, by Arts. 6 and 14,

$$a=\frac{AC}{t};\quad v=\frac{PQ}{t}.$$

Making these substitutions and writing for AO and PO their values v and r, respectively, we have

$$a=\frac{v}{r}\,v=\frac{v^2}{r}.$$

To determine the direction of the acceleration of velocity at P, it is to be observed that AC is always perpendicular to the direction of the chord PQ. Now when Q is indefinitely near to P, the direction of the chord PQ coincides with that of the tangent PT. Hence, ultimately AC becomes parallel to the radius PO. Therefore the direction of the acceleration of velocity, at any instant, is along the radius and towards the centre of the circle.

Hence, when the amount of velocity of a point moving in a circle is constant, the acceleration of velocity is, at any instant, directed towards the centre of the circle; it is constant in amount; directly proportional to the square of the velocity, and inversely proportional to the radius of the circle.

Cor.—When the angular velocity is given, we have (Art. 10)

$$a=r\omega^2.$$

When the periodic time is given,

$$a=\frac{4\pi^2r}{T^2}.$$

These equivalent expressions for the amount of acceleration are often useful.

Obs.—If from *any* point O a line OA be drawn representing in magnitude and direction the velocity of a moving point P at every instant, the curve traced out by the extremity A of the line is called the *hodograph* of the path which P describes. It may be easily shown, as in this article, that, in general, the velocity of the point A in the hodograph represents in magnitude and direction at any instant the acceleration of the velocity of P at the same instant in the original path.

The hodograph was so named by its inventor, the late Sir W. R. Hamilton, Professor of Astronomy in the University of Dublin.

38. CURVATURE.—When a point moves in a curve, the direction of motion, which is the direction of the tangent drawn to the path at the point, continually changes. The rate of change of the direction of the tangent at any point per unit of length of the curve is called the *curvature* at that point.

When the tangent turns through equal angles, while the moving point describes equal arcs along the path, the curvature is said to be uniform. This is the case when a point moves in a circle. For, the angle between the tangents drawn at P and Q (diagram, Art. 37) is equal to the angle POQ at the centre, and equal angles at the centre subtend equal arcs.

In the case when the curvature is uniform,—that is, when the path is a circle,—let θ be the angle, in circular measure, through which the tangent turns while the point describes any arc s, however large or small. Then (diagram, Art. 37) angle $POQ = \theta$, and arc $PQ = s$. Thus

$$\text{Curvature} = \frac{\theta}{s} = \frac{1}{r}.$$

The uniform curvature of a circle is thus measured by the reciprocal of its radius.

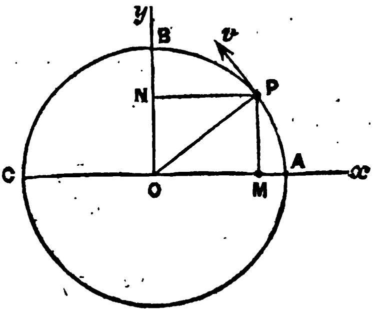
When the path is not a circle, the curvature is variable, but it may be measured, on similar principles to those explained in Arts. 6 and 14, by the above equation, when θ is the angle described by the tangent while the point describes an indefinitely small arc s, through which the curvature may be considered uniform.

The curvature at any point of a curve is thus measured by the curvature of the circle an indefinitely small arc of which coincides with an indefinitely small arc of the curve at the point. This circle is called the *circle of curvature*, at the point, and its radius is called the *radius of curvature*.

39. When a point moves in any manner along any curve we may consider, *independently*, the resolved components of its acceleration of velocity at any instant along the tangent, and perpendicular to the tangent; that is, in the direction of the normal at the point. The acceleration of velocity along the tangent is the acceleration of the amount of velocity, and may be calculated, when the circumstances are given, by the methods already explained. The acceleration along the normal, which is due to the change of direction of velocity only, is equal to the acceleration of a point moving in the circle of curvature at the point with a uniform velocity equal to the velocity of the point at the instant under consideration.

The resultant of these resolved components, which is the actual acceleration of the point's velocity, can be found by the methods already described.

40. The following analytical proof of the important results of Art. 37 will be useful to the student as an example of general principles and methods.



We shall estimate the component velocities of the point parallel to rectangular directions Ox and Oy through the centre of the circle.

Let P be the position of the moving point at the end of any time t, reckoned from the instant when P was on Ox; v_1 and v_2 its resolved velocities parallel to Ox and Oy respectively, and let ω be the angular velocity of the point round the centre of the circle, so that $v = r\,\omega$ (Art. 10). Also, let angle $POA = \theta$, so that $\theta = \omega\,t$ (Art. 9).

Now the resolved components of the velocity of P are found by multiplying v by the cosines of the angles, which the tangent at P makes with lines through P parallel to Ox and Oy. Thus

$$v_1 = -v \sin\theta = -v \sin \omega t;$$
$$v_2 = v \cos\theta = v \cos \omega t,$$

v_1 being negative, because as the time increases OM decreases. From these equations we have to find the rates at which the resolved velocities change with the time.

Suppose that t increases to $t + h$, where h is indefinitely small, and that v_1 is thus changed into u_1. Then

$$u_1 = -v \sin \omega(t + h)$$
$$= -v(\sin \omega t + h\omega \cos \omega t),$$

because h is indefinitely small, and therefore we may take $\cos\omega h = 1$ and $\sin\omega h = \omega h$, ω being in circular measure. By subtraction we get

$$u_1 - v_1 = -h\omega v \cos \omega t.$$

Therefore $$\frac{u_1 - v_1}{h} = -\omega v \cos\omega t = -\frac{v^2}{r} \cos\theta.$$

But $u_1 - v_1$ is the change of the resolved velocity parallel to Ox in the time h, and therefore $\frac{u_1 - v_1}{h}$ is the resolved acceleration, when the time h is indefinitely small. Hence, if a be the resolved acceleration of velocity parallel to Ox, we have

$$a_1 = -\frac{v^2}{r} \cos\theta.$$

Similarly it may be shown, if a_2 be the resolved acceleration parallel to Oy, that

$$a_2 = -\frac{v^2}{r} \sin\theta.$$

Let a be the resultant acceleration and ϕ the angle which its direction makes with Ox. Then (Art. 36),

$$a = \frac{v^2}{r}; \quad \tan\phi = \tan\theta.$$

The second of these equations shows that the acceleration of the point's velocity is at any instant directed towards the centre of the circle.

41. HARMONIC MOTION.—When a point P moves uniformly round a circle (Fig., Art. 40), the foot M of the perpendicular drawn from P on a fixed diameter AC of the circle is said to execute a *simple harmonic motion.*

As P moves from A round the circle, in the direction ABC, M passes from A through O to C and then back from C through O to A. The time which elapses from any instant until M moves again in the same direction through the same position is called the time of a complete oscillation, or the *period* of the harmonic motion, and is evidently the same as the periodic time of the point P in the circle. The range OA, or OC, on either side of the middle point of the course is called the *amplitude.*

By definition, Art. 29, the velocity and acceleration of velocity of M are the same at any instant as the resolved velocity and resolved acceleration of velocity of the point P in the *auxiliary* circle. Thus at A and at C the velocity of M is zero, and at O its velocity is the same as the velocity of the point P on the auxiliary circle.

The acceleration of the velocity of M is always directed towards O; let a_1 be its value, at any instant, when $OM = x$. Thus

$$a_1 = -\frac{v^2}{r}\cos\theta = -\frac{v^2}{r}\frac{x}{r}.$$

The negative sign indicating that the acceleration of velocity is towards O, and is therefore a retardation of velocity in the positive direction.

Let the constant quantity $\frac{v^2}{r} = \mu$

Then $$a_1 = -\mu x$$

Hence, when a point executes a simple harmonic motion, the acceleration of its velocity is directly proportional to the distance from the middle point of the path.

μ is the acceleration of velocity at the unit of distance, and is also called the *absolute* acceleration.

Conversely, when a point moves in a right line in such a manner that its acceleration is always directed to a fixed point on the line, and proportional to the distance from the fixed point, the moving point executes a simple harmonic motion.

To find the period of the harmonic motion, since it is the same as the period T in the auxiliary circle, we have

$$\mu = \frac{v^2}{r^2} = \frac{4\pi^2}{T^2}.$$

Therefore

$$T = \frac{2\pi}{\sqrt{\mu}}.$$

The period of a simple harmonic motion thus depends on the acceleration of velocity at the unit of distance, but is independent of the amplitude.

Obs.—The name "simple harmonic motion" is given from the fact that the simple vibrations of sounding bodies are of the nature here described. The reader who wishes for further information on this subject (and on all such subjects) is recommended to consult Sir W. Thomson and Professor Tait's treatise on Natural Philosophy.

42. RELATIVE MOTION.—Let A be a point moving in any manner in a plane (Fig., Art. 30), and suppose a second point P also to move in any manner in the same plane. Then as A and P move M, and m also move along Ox, and the distance Mm will in general change with the time; likewise Nn will change with the time.

If we supposed A to be at rest and Mm and Nn to change as they actually do change with the time, the curve which would thus be described by P is called the *relative* path of P with respect to A. Or, we may define the relative path of P about A to be the curve which would be described by P if A were at rest and if AP were to change in magnitude and direction as it actually does change when both points are in motion, the change of direction of *AP* being measured by the angles which it makes with some fixed direction. For example, if the length *AP* remained constant while *A* and *P* moved, but while the direction of the line *AP* varied, the relative path of *P* about *A* would be a circle (or a portion of a circle) and this circle would be described by *P* round *A*, if *A* were to remain at rest and *Mm* and *Nn* to change with the time, as they actually do change, so as to preserve the distance *AP* constant.

The velocity of a point moving in the relative path, as *P* would move under the supposed circumstances, is the *relative* velocity of *P* with respect to *A*, and the acceleration of the velocity of such a point is the *relative* acceleration of the velocity of *P* with respect to *A*.

The rates, at any instant, at which *Mm* and *Nn* are changing, are, respectively, the components of the relative velocity parallel to *Ox* and *Oy*.

It is evident that the rate at which *Mm* is changing at any instant will be unaltered if we suppose the velocities of *M* and *m* to be each increased, or decreased, by the same amount of velocity in the same direction along *Ox;* and similarly for *Nn*.

Now if we suppose the velocities of *M* and *m* to be each changed by an amount equal in magnitude and opposite in direction to the velocity of *M* at the instant; and, likewise, the velocities of *N* and *n* to be each changed by an amount equal in magnitude and opposite in direction to that of *N* at the instant; which is equivalent to supposing the actual velocities of *A* and *P* to be each changed by an amount equal in magnitude, but opposite in direction, to the velocity of *A*, the point *A* will be reduced to rest, while the rates at which *Mm* and *Nn* change remain unaltered.

Hence the relative velocity of any point *P* with respect to another point *A* is at any instant a velocity one of whose components is the actual velocity of *P*, and the other a velocity equal in magnitude, but opposite in direction, to the actual velocity of *A*.

Thus if from any point O (Fig., Art. 31) OA be drawn to represent the velocity of a point, and if OB be drawn to represent a velocity equal in magnitude and opposite in direction to that of a second point at the same instant, then, completing the parallelogram $OACB$, OC will represent in magnitude and direction the relative velocity of the first point with respect to the second point.

Conversely, when the absolute velocity of the second point is given, and the relative velocity of the first point with respect to the second, the absolute velocity of the first can be found.

Thus if OC represent the relative velocity of the first point with respect to the second point, and OB a velocity equal and opposite to that of the second, then BC, or OA, will represent the actual velocity of the first.

In general when any number of points move in one plane the circumstances of the relative motions, with respect to one, of all the rest may be found by *changing* the velocity and the acceleration of velocity of every point by amounts equal in magnitude and opposite in direction to the velocity and acceleration of velocity, respectively of that one. Also, if the circumstances of the relative motions with respect to one, of all the rest be given and the absolute motion of that one, the absolute motions of all can be determined.

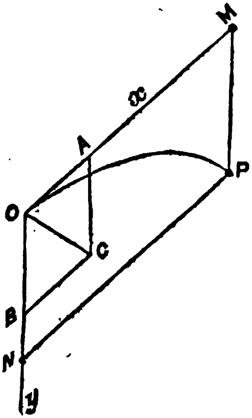
Obs.—The ordinary method of representing the motion of a point with reference to fixed axis is a simple example of relative motion. Thus, the motion of D (Fig., Art. 28) is represented by the absolute motion of M along Ox and the relative motion of D with respect to M. The student will have no difficulty in verifying that the construction of this article for the absolute velocity (and for the absolute acceleration of velocity) of D leads to the same result as the ordinary construction by the Parallelogram of velocities.

We shall now give solutions of two problems illustrating the principles and methods which have been explained.

43. Ex.—*A point moves in such a manner that its velocity parallel to a given direction is uniformly accelerated; given the magnitude and direction of its velocity at any instant to determine the circumstances of its motion.*

Let O be the position of the point at the given instant; draw Ox in the direction of its velocity at that instant, then Ox is a tangent to the path at O. Through O draw Oy parallel to the direction in which the point's velocity is accelerated.

To determine the position of the point at any time t reckoned from the given instant, let V be the velocity of the point at the commencement of the time, and let a be the amount of acceleration of velocity parallel to Oy.



Now, if we estimate the velocity of the point by its component velocities parallel to Ox and Oy, the component velocity parallel to Oy will be uniformly accelerated and the component velocity parallel to Ox will be constant. Hence, if we determine M on Ox and N on Oy from the equations

$$OM = Vt, \quad ON = \tfrac{1}{2} a t^2,$$

and draw parallels to Oy and Ox, respectively meeting in P (or, what is the same thing, measure MP from M parallel to Oy and equal to ON), the position of the point at the end of the time t is determined.

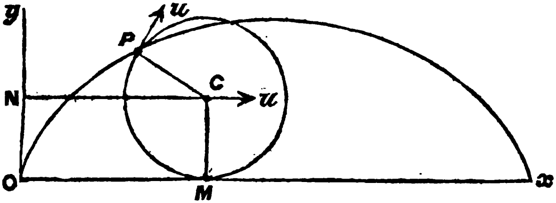
To find the magnitude and direction of velocity at the end of the time t; take OA along Ox to represent V, and OB along Oy to represent the change of velocity, at, parallel to Oy in the time t; complete the parallelogram $OACB$. Then the velocity at P is parallel to OC, and is represented in magnitude by OC. Thus the circumstances of the motion of the point are completely determined.

Cor.—By eliminating t from the above equations we get

$$OM^2 = PN^2 = \frac{2V^2}{a} ON.$$

A relation which must be true for all values of t and therefore for all points on the curve. The curve which possesses this property—that the square of the ordinate is proportional to the abscissa—is a *parabola.* The axis of the parabola is parallel to Oy. The directrix is perpendicular to yO produced, at a distance $\frac{V^2}{2a}$ from O; the focus is at an equal distance from O, on a line making with Ox an angle equal to the supplement of the angle yOx. These results follow from the geometrical properties of the parabola, and the curve can be geometrically constructed when the velocity at any point is given, in magnitude and direction, and also the amount of acceleration of velocity parallel to the given direction.

44. Ex.—*A circle rolls along a straight line, its centre moving with uniform velocity; to determine the motion of a point on the circumference of the circle*



Let C be the position of the centre of the circle at any instant, Ox the fixed line on which the circle rolls, P the position of the point in the circumference (fixed with reference to the circle) whose motion is to be determined. Let the time be reckoned from the instant when P coincides with the point O on Ox, and let Ox and Oy, perpendicular to Ox, be taken for lines of reference.

Through C draw CM and CN perpendicular to Ox and Oy; let u be the velocity of the centre C, which moves along the fixed line CN; θ the angle PCM described by the radius PC in the time t, measured from the line CM which is always parallel to Oy; and r the radius of the circle.

As the circle rolls every point along the arc PM comes successively in contact with the line Ox. Hence,

$$\text{Arc } PM = OM = ut.$$

Therefore $$r\,\theta = ut,$$

or $$\theta = \frac{u}{r}t.$$

Hence the angular velocity of P about C is uniform, and if ω be its value, we have

$$\omega = \frac{u}{r}; \quad u = r\omega.$$

But $r\omega$ is the velocity of P round C. Hence the velocity of P in its relative circular path is the same as the velocity of the centre C along NC. Thus we are given the absolute motion of C and the relative motion of P with respect to C and it is required to determine the absolute motion of P.

Let v_1 and v_2 be the resolved components of the absolute velocity of P parallel to Ox and Oy respectively; and let us suppose the velocities of P and C to be each changed by an amount equal and opposite to that of C. Thus C will be reduced to rest and the relative velocity of C unaltered (Art. 42). Thus the resolved components of the relative velocity of P must be equal to $v_1 - u$ and v_2. Equating these values to the resolved components of the relative velocity u of P along the tangent at P, we get

$$v_1 - u = u \sin\left(\theta - \frac{\pi}{2}\right) = -u \cos\theta$$

$$v_2 = u \cos\left(\theta - \frac{\pi}{2}\right) = u \sin\theta$$

Let v be the absolute velocity of P and ϕ the angle which its direction makes with Ox. Then, by Art. 36,

$$v^2 = v_1^2 + v_2^2 = u^2 \left\{(1 - \cos\theta)^2 + \sin^2\theta\right\};$$

$$\tan\phi = \frac{v_2}{v_1} = \frac{\sin\theta}{1 - \cos\theta}.$$

Therefore $$v = 2u \sin \tfrac{1}{2}\theta,$$
and $$\tan\phi = \cot \tfrac{1}{2}\theta,$$
where $$\theta = \frac{u}{r}\, t.$$

These equations determine the direction and magnitude of the absolute velocity of P at any instant.

To determine the position of P at any instant, let its co-ordinates be x and y. Then

$$x = NC - PC \cos PCN = ut - a \sin\theta$$
$$y = CM + PC \sin PCN = a(1 - \cos\theta)$$

where θ has the value given above.

Cor.—The velocity of P when at the greatest distance from Ox is found, by putting $\theta = \pi$, to be $2u$, and its direction is parallel to Ox. Also the velocity of P when on Ox is found, by putting θ equal to zero (or any multiple of two right angles) to be zero.

Obs.—The curve which P describes is called a *cycloid.* The rolling circle is called the *generating* circle, and the line on which it rolls, the *directrix.*

From the equation which determines the direction of the velocity in the cycloid at any point we get the following construction for drawing a tangent to a cycloid at any point P. Describe the generating circle through P, and bisect the angle PCM; the tangent at P is parallel to the bisecting line. This construction also follows from the fact proved above that the point M of the circle in contact with Ox is instantaneously at rest, and the direction of the motion of P at any instant is therefore perpendicular to the line MP.

EXAMPLES.

1. Two points start from the same point and move uniformly along straight lines inclined at an angle θ; if the velocity of one be u, and the velocity of the other v, show that their distance apart at the end of n seconds is
$$n\,(u^2 + v^2 - 2uv\cos\theta)^{\frac{1}{2}}.$$

2. A man can pull a boat with twice the velocity of the stream; at what angle to the stream must the boat be rowed in order that he may land at a point directly opposite to his starting point?

Ans. 60°.

3. A man sets out to swim directly across a river whose breadth is b; when he arrives at the other side, he is at a distance a below the point from which he started; show that he will regain this point if he incline his course back again at an angle θ, such that

$$\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=\frac{a}{b}.$$

4. A person, travelling eastward at the rate of 4 miles an hour, observes that the wind seems to blow directly from the north; on doubling his speed the wind appears to blow from the north-east; determine the direction of the wind and its velocity.

Ans. N. W.; $4\sqrt{2}$ miles.

5. Give a construction for representing the velocity of a point B relatively to another A, the velocities of A and B relatively to a third point C being given.

6. Two points are moving with uniform velocities in two straight lines in the same plane; show that the relative path of either point with respect to the other is a straight line described with uniform velocity.

7. A point A moves along a right line with uniform velocity; a point B (which at no time is coincident with A) moves with uniform velocity along another line in the same plane with the former; show that any point which divides the line AB into parts in a constant ratio, moves along a right line.

8. If a be the distance at any time between two points which are moving uniformly in straight lines in one plane, V their relative velocity, u and v the resolved components of V, at the instant, along and perpendicular to the line joining the points; show that their distance when nearest to each other is $\frac{au}{V}$, and that the time of arriving at this nearest distance is $\frac{au}{V^2}$.

9. The components of the velocity of a point, parallel to two given directions, are uniformly accelerated; show that the path which the point describes is, in general, a parabola. If at any instant the component velocities be proportional to the component accelerations of velocity, show that the path is rectilinear.

10. If two points move in two straight lines, in the same plane, with uniformly accelerated velocites, the path of either relative to the other will be rectilinear, if at any instant their

velocities be to one another, respectively, as their accelerations of velocity. If this condition is not satisfied what will the relative path be?

11. What are the velocities, in feet per second, of the highest and lowest points of a carriage wheel of radius two feet, as it rolls along at the rate of six miles an hour, (1) relatively to the centre of the wheel, (2) relatively to the ground?

Ans. (1) 8·8; 8·8.
(2) 17·6; 0.

12. The highest point of the wheel of a carriage, rolling on a horizontal road, movestwice as fast as each of two points in the rim, whose distance from the ground is half the radius of wheel.

13. If the velocity of a point be uniform in amount, and if the direction of motion revolve with uniform angular velocity, show that the path is a circle.

14. If the velocity of a point moving in a plane curve vary as the radius of curvature, show that the direction of motion revolves with uniform angular velocity.

15. A line of given length moves with its extremities in two given lines which intersect; show how to draw a tangent to the path described by any point on the line.

16. A straight line moves in any manner in a plane; prove that, in general, at any instant, the directions of motion of all its points are tangents to a parabola.

KINETICS.

CHAPTER I.

DYNAMICAL LAWS AND PRINCIPLES.

1. In the preceding chapters on Kinematics, we have considered the circumstances of the motion of a moving point, and explained the general methods by which, from the laws which regulate the change of velocity (supposed to be given), the position and velocity of the point can be determined, at any time, when its position and velocity at any other time are given. Since a material particle, as far as its position is concerned, may be regarded as a point, these investigations will also apply to the motion of a material particle, when the laws which regulate the change of the velocity of the particle are given. The motion of a rigid body may also be fully represented by the motion of a point, when, during the motion, the parts of the rigid body preserve the same relative positions amongst each other; that is, when the body does not rotate. In general, however, in the motion of a rigid body, different parts move differently, and the motion cannot be determined from the motion of a single point.

The kinematical results which have been arrived at are altogether independent of the ideas of *force* and *matter*. However, in the material world, any case of motion must be the motion of a body of definite size and mass, and as we can only distinguish the state of rest or motion of a body by comparing it with other bodies, all motions which can come under our observation must be relative. We have now to consider the actual causes which do produce motion from relative rest, or which produce a change of relative motion.

Force has been defined as that which produces (or which would produce, if uncounteracted) a change in a body's state whether of relative rest or of relative motion, and what is meant by a change of motion, in the case of a body of definite mass, will be explained further on. We have already explained (*Kinematics*, Art. 32) what is meant by the change of velocity of a moving point, and how it is to be measured.

The object of the science of *Kinetics* is to investigate the circumstances of the motion which result from the action of given forces on a given body, and, conversely, to investigate the nature of the forces which would maintain given circumstances of motion in a given body.

Our investigations will be confined to simple cases of the motion of a material particle (or cases when the motion of a rigid body may be represented by that of a particle); that is, to the *Kinetics of a particle*, the *Kinetics of a rigid body* not being sufficiently simple to allow of its introduction into a treatise so elementary as the present, and moreover requiring for its development a more extensive knowledge of mathematics than that which the student is assumed to have acquired.

Obs.—Kinetics includes, as a particular and much simpler case, the science of Statics—when the forces are such as to counteract each other's effects so that no change of motion is produced in the body on which they act (*Statics*, Art. 6), and it will be seen that the definitions and principles given in the Introduction to Statics, Chapter I (which should be again read and considered in connection with the present subject), are included under the fundamental principles of Kinetics now to be explained.

2. MOMENTUM.—The *quantity of motion*, or the *momentum* of a body, which is moving without rotation, is defined to be, at any instant, conjointly proportional to the mass (quantity of matter) of the body, and the common velocity with which all its parts are moving.

Hence, the momentum of a body, whose mass is m, moving with a velocity v, is proportional to mv; and if we take as unit of momentum the momentum of the unit of mass moving with the unit of velocity, we have, in general,

$$\text{momentum} = mv.$$

The change of the quantity of motion, or the change of momentum, of a body is equal to the product of the mass moving and the change of velocity in any time. Thus, when the change of momentum takes place in the direction of motion,—that is, when the direction of motion does not alter,—the change of momentum is the product of the mass moving and the increase or decrease of the amount of velocity. But when the direction of motion alters,—that is, when the motion is not rectilinear,—the change of momentum is the product of the mass of the body and the change of velocity estimated as explained in *Kinematics*, Art. 32. Thus if from any point O, OA and OC be drawn, representing, respectively, the momentum of a body at the commencement and end of any interval of time, then AC, or OB, represents in magnitude and direction the change of momentum in the interval.

The rate of change of momentum or the acceleration of momentum is the product of the mass of the body moving and the acceleration of its velocity at the instant, and the direction of acceleration of momentum is the same as the direction of the acceleration of velocity.

Obs. 1.—When a body moves without rotation all its points describe paths identical in form, but not in situation, and are moving at any instant with equal velocities in parallel directions. Thus we speak of the velocity of a body, whose motion consists of a motion of translation only, as the common velocity with which all its points are moving, and the motion of any one point determines those of the others. This is the kind of motion for which momentum and change of momentum have been defined above.

Obs. 2.—We have assumed that the momentum of a body may be represented by a straight line; this is evident, since the momentum is measured by the product of the mass of the body and its velocity, and as long as we confine our attention to the same body, the lines which represent momenta will be proportional to the lines which represent the corresponding velocities.

3. When a body has a motion of rotation as well as a motion of translation, the momentum of the body resolved

in any direction is the sum of the resolved parts of the momenta of the particles, which compose the body, in the same direction. So, the change of momentum resolved in any direction is the sum of the resolved parts of the changes of momentum of the particles.

If we adopt the same notation as in *Statics*, Art. 113, and suppose the body to be composed of particles, we have for one of the co-ordinates of the centre of inertia (centre of gravity)

$$\bar{x}\,\Sigma m = \Sigma m x.$$

Suppose $\bar{x}$ changes into $\bar{x} + \bar{a}$ in an indefinitely small time t, while x (the co-ordinate of any one of the particles whose mass is m) changes into $x + a$. Then

$$(\bar{x} + \bar{a})\,\Sigma m = \Sigma m\,(x + a) = \Sigma m x + \Sigma m a$$

subtracting from this the equation given above and dividing by t, we get

$$\frac{\bar{a}}{t}\,\Sigma m = \Sigma m \frac{a}{t}.$$

Hence, if $\bar{v}$ be the resolved velocity of the centre of inertia, and v the resolved velocity of the particle whose mass is m,

$$\bar{v}\,\Sigma m = \Sigma m v \text{..................} \quad (1)$$

Thus the momentum of a body resolved in any direction is equal to the momentum resolved in the same direction of a mass equal to the mass of the body moving with a velocity equal to the velocity of the centre of inertia.

By exactly similar reasoning from equation (1) it may be shown, if $\bar{\alpha}$ be the resolved acceleration of velocity of the centre of inertia, and α the resolved acceleration of velocity of the particle whose mass is m, that

$$\bar{\alpha}\,\Sigma m = \Sigma m \alpha. \text{ ....} \quad (2)$$

Thus the acceleration of momentum of a body resolved in any direction is equal to the acceleration of momentum resolved in the same direction, of a mass equal to the mass of the body, and moving with an acceleration of velocity equal to the acceleration of velocity of the centre of inertia.

Cor.—When the velocities and the accelerations of velocity of any number of particles of given masses (whether connected with one another or detached) are given, the velocity and acceleration of velocity of the centre of inertia can be found. Let the velocities and accelerations of velocity be estimated by their resolved components parallel to three directions in space Ox, Oy, Oz, mutually at right angles. The equation (1) of this article gives the velocity of the centre of inertia parallel to Ox, and two similar equations give its resolved velocities parallel to Oy and Oz. Also equation (2) gives the acceleration of velocity of the centre of inertia parallel to Ox, and two similar equations give the accelerations of its velocity parallel to the other axes.

4. The three following laws, which are given as they were enunciated by Newton in his *Principia* nearly two centuries ago, form a complete foundation for the Static, as well as the Kinetic, branch of Dynamics. They are usually called *Newton's Laws of Motion.* The first two were given by Galileo, who was the first to lay down accurate notions on the subject, and the third was also known before the publication of the *Principia.*

These laws are not to be regarded as in any sense self-evident, nor are they even readily admissible when first enunciated. They are the results of the accumulated experience and observation of many minds, and the history of Dynamical science bears evidence to the great difficulties which lay in the way of forming correct notions concerning the action of force in producing motion in matter. Even the Greeks, who excelled in Geometry and also made some progress in Statics, completely failed in their attempts to solve problems of motion; and the honor of establishing the fundamental principles of Kinetics was reserved for Galileo.

The difficulty of forming correct notions regarding kinetical laws arises from the complicated nature of the forces which are brought into action in all cases of motion which come under our observation. The fact that all bodies which we observe on the earth are ultimately reduced to rest by the action of such resisting forces as friction and the resistance of the air, and that heavy bodies fall (in air, not in a vacuum) more rapidly than light bodies, would at first sight (the effect of the resisting forces not being taken into consideration) suggest erroneous conceptions on the subject.

The same difficulty is encountered in making kinetical laws the object of *direct* experiment. We cannot study the motion produced by a given force in a given body, because such disturbing forces as friction and the resistance of the air, which always come into action when motion takes place, can never be wholly eliminated, and the amount of change of motion produced by such forces is not easily estimated.

Further on experiments will be described which, when disturbing causes are, as far as possible, eliminated or accounted for, verify the laws now to be given. But the most convincing argument in their favour is the complete agreement of calculations founded on them with observed cases of motion, especially the agreement of the positions of the bodies of the Solar System, calculated on the assumption of the truth of these laws, and the law of gravitation, with their positions as actually observed.

5. LAW I.—*Every body continues in its state of rest or of uniform rectilinear motion, except in so far as it may be compelled by impressed forces to change that state.*

This law asserts that matter has no power in itself to move itself, or to *change* in any way its own motion, either in amount or direction, when once it is in motion. But matter possesses an innate power of resistance so that some external influence is requisite to change its natural state whether of rest or of uniform motion in a straight line, and *force* is thus defined as that which produces or which tends to produce such a change (*Statics*, Art. 4).

The property in virtue of which matter resists change of motion is called the *inertia* of matter, and the force of resistance is called the *force of inertia.*

It will be explained further on that every body in the universe is acted on by external force, so that it is impossible to observe any case of motion in which a body is not acted on by force. When, however, the forces acting on a body are in equilibrium, its condition is the same as if it were acted on by no force whatsoever, and the body will either remain at rest, or, if once in motion, its motion will be uniform (in magnitude and direction) as long as all the forces acting on it continue in equilibrium.

Obs.—The motion here contemplated is a motion of translation. In order to exclude the idea of rotation, the laws of motion are often enunciated for a particle.

6. With regard to the arguments in favour of the truth of this law, we may remark that the more nearly the condition of the forces acting on a body being in equilibrium is satisfied, the more nearly is the body's motion found to be uniform. Thus a stone thrown along a horizontal plane is retarded by friction and the resistance of the air, and it is soon deprived of its motion by these resisting and unbalanced forces. The smoother however is the plane horizontal surface, along which the stone is projected, the further will it move. For example, a stone thrown along the (comparatively) smooth surface of a sheet of ice, moves further, and hence more nearly uniformly, than when the same stone is thrown along a level road, because the friction against the ice is less than that against the road. Direct experiments will be noticed further on, in which the condition of the forces acting on the body being in equilibrium is as nearly as possible satisfied; and in such cases motion is observed to be practically uniform. However, as before stated, the complete agreement of calculation with observation renders the truth of this and the other laws of motion, as certain as any inductive knowledge can be.

Obs.—Absolute rest, and hence absolute velocity, are nowhere discoverable in nature. The motions of bodies on the earth can be estimated relatively to it, but the earth has not only a diurnal motion of rotation, but an annual motion in its orbit round the sun, and the sun moves relatively to the so-called fixed stars, which may also be in motion relatively to other bodies in the universe. Hence, a body's state of rest or motion is to be regarded as relative to the common motion of surrounding bodies which are relatively at rest amongst themselves. We have now to consider the measure of the force which produces a change of a body's state of relative rest or relative uniform motion. This leads to the enunciation of the second law.

7. LAW II.—*The change in the quantity of motion is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.*

By the change of the quantity of motion is here to be understood the change of momentum (Arts. 2 and 3), and the law asserts that the amount of change of momentum is proportional to the force which produces it, and that the direction of the change of momentum is the direction in which the force acts, whether the body be originally at rest, or moving with any velocity, or be acted on simultaneously by any forces. Hence Newton's second law is equivalent to the following statements:—

(*a*). *If any number of forces act on a body, whether originally at rest, or moving in any manner, each force produces the exact change of motion which it would have produced if it had acted singly on the body at rest.*

(*b*). *Forces are proportional to the changes of momenta which they would generate in a given time.*

Obs.—It has been the custom with many writers on Dynamics to enunciate as the second law of motion the statement (*a*) given above, which is contained in Newton's second law; and their third law, which is the same as the statement (*b*), is deduced from Newton's first law and the law (*a*), by the help of particular definitions of the equality of masses and the equality of forces, which definitions also follow as immediate consequences from Newton's second law. The tendency, however, in the present day is to return to the method of Newton.

8. MEASURE OF FORCE.—When a force, which continues uniform in amount and direction, acts on a free body, it generates, by the second law of motion, equal changes of momentum, and hence equal changes of velocity, in equal intervals of time. Hence, by the action of the force, the velocity of the body is uniformly accelerated in the direction of the force. It is convenient to measure force by the momentum generated in the unit of time, which is equal to the acceleration of momentum when the force is constant.

Let P be the magnitude of a constant force acting on a body whose mass is m, and generating a change of velocity

a in the unit of time, a being thus the measure of the acceleration of velocity produced by the force. Then, by the second law of motion,

$$P = kma,$$

k being the constant of variation whose numerical value depends on the units of force and mass which are chosen.

If we take for unit of force the force which generates the unit of velocity in the unit of time in the unit of mass, we have $P = 1$ when $m = 1$ and $a = 1$, and hence $k = 1$. Thus, on this supposition,

$$P = ma.$$

Hence force, when uniform, is measured by the change of momentum which it generates (or which it would generate if uncounteracted, in the case of a body which is not free to move) in the unit of time; that is, by the acceleration of momentum which it produces.

A force is variable in amount or in direction, or in both, when it does not generate equal changes of momentum parallel to a given direction in equal times, but it is to be measured at any instant by the change of momentum which would be generated in the next unit of time if the magnitude and direction of the force were to continue uniform for that unit of time and the same as at the instant under consideration; that is, force is to be measured at any instant by the rate of change, or acceleration, of momentum which it produces in the body at the instant.

Cor.—If v be the change of velocity generated by a uniform force in a time t in a body of mass m, we have, since $v = at$,

$$P = \frac{mv}{t}.$$

The magnitude of the force is thus directly proportional to the mass moved and to the change of velocity, but inversely proportional to the time in which a given change is produced.

9. From the equation $P = ma$ we have the following consequences:—

(1). If different forces act on the same body, the forces are proportional to the accelerations of velocity which are (or would be) produced by them.

Hence, as long as we confine our investigations to the same body, forces may be measured by the accelerations of velocity which they produce, and if the mass of the body be the unit of mass, this measure of force will be the same as that given in Art. 8.

Again, two forces are equal when they would produce equal accelerations of velocity in the same body, or in the same particle, and this definition of the equality of forces evidently includes the definition given in *Statics*, Art. 7.

(2). If different forces act on bodies of different masses, and produce the same acceleration of velocity in all, the forces are proportional to the masses of the bodies.

It will be explained further on that the force of the earth's attraction, or gravity, always produces (or would produce) the same acceleration of velocity in all bodies at its surface, in the same locality, and hence the force with which a body gravitates, or the weight of the body, is proportional to its mass (*Statics*, Art. 9).

Again, the masses of two bodies are equal when equal forces would produce in the bodies equal accelerations of velocity, and this definition of equality of masses, thus includes the definition given in *Statics*, Art. 9.

(3). When the same force acts on different bodies, the accelerations produced are inversely proportional to the masses of the bodies.

(4). If any number of bodies be in motion, their relative motions will not be changed if they are all acted on by forces in parallel directions proportional to the masses of the bodies. Since the forces will produce equal accelerations of velocities in parallel directions.

Obs.—When a force acts on a body of the unit of mass, the force is measured numerically by the acceleration of velocity which it produces. It was the custom for this reason to call the acceleration of velocity "accelerating force." The term, however, is very objectionable, and has been the cause of much confusion. It is now, very generally, being abandoned.

10. UNITS OF MASS AND FORCE.—The units of time and velocity, and hence the unit of acceleration of velocity, being

supposed to be given, the unit of mass and the unit of force are connected by the equation $P = ma$, so that if one of the units be given the other is determined. Thus:—

The unit of force is that force which produces the unit of acceleration of velocity in the unit of mass.

The unit of mass is the mass of that body in which the unit of force produces the unit of acceleration of velocity.

The most convenient method, both for practical and scientific purposes, is to adopt some convenient national or international standard of mass and to derive from it the unit of force. Thus we might take for unit of mass the British imperial avoirdupois pound, the French gramme or kilogramme, or the Indian sér, or any convenient multiple or sub-multiple of any of these units. The unit of force is then defined to be:—

That force which, acting on the standard unit of matter during the unit of time, generates the unit of velocity.

This method of measuring force, which thus follows from Newton's second law, but which was first practically introduced by Gauss, furnishes a standard of force independent of the varying amounts of gravity in different localities. The standard of mass adopted in this book is the pound avoirdupois. (See also *Statics*, Art. 10.)

It is to be observed that the unit of force depends on the unit of mass, the unit of time, and the unit of velocity. There is thus a single reference to mass and space, but a double reference to time (*Kinematics*, Art. 11).

Obs.—The methods of measuring mass and force here given are not those which have been in common use in treatises on Dynamics, and, as far as the author is aware, they were first brought to public notice in Great Britain by the publication in 1867 of Sir William Thomson and Professor Tait's magnificent treatise on Natural Philosophy, to which work the student is referred for further information on the subject of this chapter. We shall again return to this subject in the next chapter, and the practical methods of measuring mass and force will be noticed.

11. **Parallelogram of Forces.**—Suppose two forces P and Q to act simultaneously on a particle O of given mass.

Then, by the second law of motion, each force will produce the same change of motion in its own direction that it would produce if it acted on the particle singly at rest.

Take OA in the direction of the force P to represent the rate of change of momentum, or the acceleration of momentum, which would be produced by the force P acting alone; then OA also represents the magnitude and direction of the force P. Take also OB to represent similarly the acceleration of momentum which would be produced by Q acting alone, OB also represents the force Q. Complete the parallelogram $OACB$.

Then the diagonal OC represents the acceleration of momentum which is equivalent to component accelerations of momentum represented by OA and OB (*Kinematics*, Art. 34), and is the actual acceleration of momentum when the particle is acted on by the forces P and Q simultaneously.

Hence the forces P and Q, represented by OA and OB, are equivalent to a single force R, represented by the diagonal OC of the parallelogram, described on OA and OB as adjacent sides.

Obs.—Thus the *parallelogram of forces*, which was taken as the fundamental proposition of Statics, follows as a particular case of Newton's second law.

12. KINETICS OF A PARTICLE.—When a force acts on a free particle it generates change of motion, and the acceleration of momentum at any instant is in the direction in which the force acts, and is numerically measured by the amount of the force at the instant. Hence, to determine the circumstances of the motion of a particle of given mass acted on by a given resultant force, or by a force which changes according to known laws, is the same as the Kinematical problem to determine the motion of a point whose velocity changes according to known laws.

Again, every case of the motion of a point in Kinematics, in which there is acceleration of velocity, corresponds to a similar case of the motion of a particle in Kinetics, the particle being acted on by a force at every instant in the direction in which its velocity is accelerated, and which is numerically measured by the product of the acceleration of velocity and the mass of the particle. The following are some examples:—

(1). When a constant force P acts on a free particle from rest, or, when in motion, in the direction of motion, the velocity of the particle is uniformly accelerated in a right line, the acceleration of velocity being given by the equation $P=ma$, and the motion may, hence, be completely determined by the equations given in *Kinematics*, Art. 18.

(2). A free particle acted on by a constant force always parallel to a given direction in general describes a parabola, and the motion can be completely determined as in *Kinematics*, Art. 43, but when the motion is at any instant in the direction of the force, it describes a right line, (1).

(3). A particle moving in a circle with uniform velocity is acted on by a constant force $\frac{mv^2}{r}$ directed at every instant towards the centre of the circle, *Kinematics*, Art. 37.

(4). When a particle moves in a plane curve the resultant force acting on it may be resolved into two components, at any instant; one along the tangent, which changes the amount of velocity; the other along the normal, which changes the direction of velocity only. The resolved component along the normal is at any instant $\frac{mv^2}{r}$, where m is the mass of the particle v the amount of its velocity and r the radius of curvature of the path at the position which the particle occupies at the instant, *Kinematics*, Art. 39.

(5). The motion of a particle acted on by a force proportional to its distance from a fixed point in the line of motion is a simple harmonic motion, whose period is $\frac{2\pi}{\sqrt{\mu}}$, μ being the amount of the force on a particle of the unit of mass at the unit of distance, *Kinematics*, Art. 41.

Obs.—The principles which have been already explained are sufficient for the determination of the motion of a single particle (or of a rigid body moving without rotation, in which case, as far as its motion is concerned, its whole mass may be regarded as condensed into one particle) subject to the action of given forces. But in more complicated cases, and when we have to take into account the mutual actions between two or more particles (or bodies), the third law of motion, which will now be given, is required.

13. Law III.—*To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed.*

The following are some of the consequences of this law:—

(1). When one body presses or draws another (whether the bodies be relatively at rest or in motion) the other also presses or draws the first with an equal force in the opposite direction.

This axiom has been freely used in Statics and includes, as pointed out in *Statics*, Art. 21, the principle of the *transmissibility of force* by a rigid body and by a flexible inextensible string.

(2). When one body attracts another from a distance (whether the bodies be relatively at rest or in motion) the other attracts the first with an equal force in the opposite direction.

(3). When the motion of one body is changed by impact on another the motion of the second is also changed by an equal amount in the opposite direction.

For, while the bodies are in contact the forces between them are equal and oppositely directed, and hence generate changes of momentum in the bodies equal in amount and opposite in direction, the bodies being supposed to have no motion of rotation either before or after the impact.

(4). The force of inertia of a body or its resistance to change of velocity is, at every instant, equal and opposite to the resultant force which is producing the change of velocity.

From this it follows, by the second law, that the force of inertia is directly proportional to the mass of the body.

Again, the forces acting on a particle in motion (which must be equivalent to a single resultant) are at every instant in equilibrium with the force of inertia of the particle, measured by ma, and acting in a direction opposite to the direction of the acceleration of the particle's velocity.

For example, when a particle moves uniformly in a circle the forces acting on the particle—which must be equivalent to a single resultant directed towards the centre of the circle, Art. 12, (3)—are at each instant in equilibrium with the force of inertia $\frac{mv^2}{r}$ acting in the direction of the radius and from the centre of the circle.

Obs.—The force of inertia of a particle moving uniformly in a circle, or, more generally, the resolved component of the force of inertia along the normal to the path, is usually called the *centrifugal force.* It should be carefully remembered that the force of inertia is a purely *resisting* force, and has no existence except simultaneously with acceleration of velocity.

14. D'ALEMBERT'S PRINCIPLE.—Every rigid body may be supposed to be made up of indefinitely small particles, each particle being held in its position relatively to the others by the forces exerted upon it by the surrounding particles. Hence, Art. 13 (4), the force of inertia of each particle (its reaction against acceleration of velocity) must be equal and opposite to the resultant of *all* the forces which act on it, arising both from the actions of other particles of the body and from external forces. Thus the force of inertia of each particle is in equilibrium with all the forces acting on that particle. This is true for all the particles of which the rigid body is composed. Hence, by the principle of superposition of forces in equilibrium, all the forces, internal and external, which act on the particles of a body are in equilibrium at every instant with the forces of inertia (arising from acceleration of velocity) of all the particles.

Now the forces which surrounding particles exert on any one particle are, by the third law, equal and opposite to the forces which that particle exerts on those surrounding particles; and this being true for all the particles of which the body is composed, the internal forces must in the aggregate form a system of forces in equilibrium amongst themselves. Hence:—

The forces of inertia of all the particles of a body (the reactions against accelerations of velocity) are at every instant in equilibrium with the external forces acting on the body.

Obs. 1.—This is the celebrated principle first explicitly stated by D'Alembert and known by his name. That it is implied in Newton's interpretation of his third law was first pointed out (as far as the author is aware) in Sir W. Thomson and Professor Tait's Natural Philosophy.

Obs. 2.—The Kinetics of a rigid body is usually founded on D'Alembert's principle. We shall now use it to establish the very important proposition of the next article, by which the motion of translation of a rigid body may be represented by the motion of a particle.

14. *The motion of the centre of inertia of a rigid body is the same as the motion of a particle having a mass equal to the whole mass of the body and acted on by forces equal and parallel to those which act on the body.*

We shall estimate the forces which act on the different particles of the body and the accelerations of the velocities of the particles by their resolved components parallel to three fixed directions in space Ox, Oy, Oz, mutually at right angles.

Let m be the mass of any particle, X the resolved component of the external resultant force acting on it, and a the resolved component of its acceleration of velocity, parallel to Ox.

Then, by D'Alembert's principle, the group of parallel forces denoted by ΣX and $-\Sigma ma$ must be in equilibrium. Hence

$$\Sigma X - \Sigma ma = 0.$$

But, Art, 3, $\qquad \Sigma ma = \quad a\,\Sigma m.$

Therefore $\qquad \Sigma X = \quad a\,\Sigma m.$

Thus the acceleration of the velocity of the centre of inertia, resolved parallel to Ox, is the acceleration of velocity which would be produced in a mass Σm by a force ΣX parallel to the direction Ox. The same thing may similarly be shown to be the case for the resolved accelerations of velocity of the centre of inertia parallel to Oy and Oz; which proves the proposition.

Cor. 1.—Hence it follows, from the first law of motion, that the centre of inertia of a rigid body, free from the action of external forces, is either relatively at rest or moves uniformly in a right line.

Cor. 2.—If a system of bodies be subject only to their mutual actions and reactions, their centre of inertia is either at rest or moves uniformly in a right line.

For example, the centre of inertia of the solar system is either at rest or moves uniformly in a right line, if we suppose the, so-called, fixed stars to be so far away as not to exert any appreciable attraction,

Obs.—The proposition of this article is usually called "the principle of the conservation of the motion of the centre of inertia." We shall return to some important consequences of the third law in the Chapter on Energy.

CHAPTER II.

THE FORCE OF GRAVITY.

16. Falling Bodies.—The following are the laws which have been observed to regulate the motion of bodies which are allowed to fall freely to the surface of the earth (in the same locality) from heights above it which are inconsiderable compared to the earth's radius.

(1). Bodies always fall in vertical directions.

(2). The velocity of a falling body is uniformly accelerated.

(3). The amount of acceleration of velocity is the same for all bodies, whatever be their masses.

These laws are only approximately true for bodies falling in air, because the resistance of the air disturbs and changes their motion, but if experiments be conducted in a vacuum:—

The first may be shown to be true by allowing a series of bodies to fall. They will be found always to fall vertically.

The second may be verified by allowing the same body to fall from different heights. It will be found that the heights are proportional to the squares of the times of falling, and hence, *Kinematics*, Art. 19, the acceleration of velocity is constant or uniform.

The third may be verified by allowing any number of bodies of different masses and composed of different materials to fall from the same height at the same instant. They will be found all to reach the ground at the same time, and hence the acceleration of velocity must be the same for all. Thus a feather and a piece of lead when let fall from the same height, at the same instant, in a vacuum, will reach the ground in the same time. The feather, however, would fall slower in air than the lead, because the resistance of the air on the feather is greater in proportion to its weight than the resistance of the air on the lead.

17. The Force of Gravity.—The force which causes a free body to fall with accelerated velocity towards the earth

is called the *force of gravity* on the body, or the *weight* of the body, and when a body rests on a horizontal plane it is equal to the resultant pressure of the body on the plane. From the observed laws of falling bodies, given in the last article, and Newton's second law of motion, we have the following consequences:—

From (1): the direction of the force of gravity is always vertical, and, hence, the gravitating forces of different bodies, at the same place, may be considered as parallel forces.

From (2): the force with which any given body gravitates is a uniform or constant force.

From (3): the gravitating force of a body, or the weight of a body, is proportional to its mass, Art. 9. (See also *Statics*, Art. 9.)

Let W be the weight of a body whose mass is m, and let g be the uniform acceleration of velocity which would be produced in the body, if allowed to fall freely, by its weight. Then, Art. 8,

$$W = mg.$$

g is a constant quantity for all bodies in the same locality, near the earth's surface. The numerical value for g is also the numerical value of the weight of the unit of mass expressed in Kinetic units of force. (See also *Statics*, Art. 10.)

Obs.—When g is called the *force of gravity* it should be remembered that what is meant is that g is the numerical value in Kinetic units of the force of gravity on the unit of mass. Methods for determining its value experimentally at any place will be given further on.

18. **Practical Measurements of Mass and Force.**—The masses of bodies are practically determined by comparing their weights in any locality with the weight of some given standard body whose mass is taken for the unit of mass, since we have seen that the mass of a body is proportional to its weight at a given place; and a balance with a set of standards of mass, which are multiples and submultiples of this unit, will serve to determine the mass of a body in any locality whatsoever. (See also *Statics*, Art. 157.)

According to the gravitation method of measuring forces, the weight of a body is expressed numerically by the same number which expresses its mass, and a spring balance correctly graduated for any locality also serves to determine masses of bodies in that locality. But a spring balance is a measure of force, not of mass, and will only determine masses correctly in the latitude for which it is graduated. For example, if a spring balance indicated masses in pounds in London, it would indicate too large a mass at the equator, because the force of gravity is less at the equator, and, hence, a greater mass would be required to produce by its weight a given extension of the spring. A perfect spring balance would however determine correctly the force with which a given mass gravitates in any locality.

It is evident that a perfect spring balance graduated in Kinetic units might be used to measure such forces as steam pressure, muscular power, &c. Practically forces are measured in gravitation units by the weights of the standards of mass which they are capable of counterpoising, when properly applied; but when these measurements are required to be strictly accurate the gravitation units must be reduced to Kinetic units, by multiplying by the values of g for the different localities where the measurements are made.

Obs. 1.—The British unit of mass—the pound avoirdupois—is the mass of a certain piece of platinum kept at the Exchequer Office in London. Standard pounds which have been accurately compared with this primary standard are kept at different places for reference and in case the original pound should at any time be destroyed. The standard pound could however be at any time reconstructed from the numerical relations which connect it with other units, artificial and natural.

Obs. 2.—It is not, of course, always necessary in order to ascertain and express numerically the masses of bodies to place them in a balance and subject them to direct comparison with standards. For example, when a body is homogeneous it is obviously sufficient, in order to determine its mass, to know its volume and the mass of the unit of volume.

19. **Density.**—When a body is such that any, the same, volume of it however small, and wherever taken within the body, always contains the same quantity of matter, the body is said to be *homogeneous*, or *of uniform density*, otherwise, it is said to be *heterogeneous*, *or of variable density.*

The density of a homogeneous body is measured by the mass contained in a unit of volume. Hence, if m be the mass of a volume V of a homogeneous body whose density is ρ, we have

$$m = \rho V$$

When the density of a body is uniform it is the same at every point in the body and its numerical value is derivable from the above equation, when m is the mass contained in a volume V of the body, however large or small the volume be taken.

In heterogeneous bodies the density is different at different points, but the density at any given point is measured by what would be the mass of a unit of volume of a homogeneous body, having a density the same as the density of the body at the point under consideration. This is, in fact, expressed by the above formula when the density is derived from the mass of an indefinitely small volume round the point, throughout which the density may be considered uniform.

Def.—The *average* or *mean* density of a body is measured by the mass of the body divided by its volume.

20. **Relative Density.**—The densities of homogeneous bodies are proportional to the masses of equal volumes of the bodies, and any convenient substance may be taken as a standard, and the relative densities of other bodies may be estimated with reference to its density. Thus the relative density of a body with reference to the density of a certain standard, substance is measured by the ratio of the mass of a given volume of the body to the mass of an equal volume of the standard substance.

Relative densities may be converted into absolute densities by multiplying the relative densities by the absolute density of the standard substance; that is, by the mass of its unit of volume.

Obs. 1.—Practical methods for determining relative densities with reference to convenient standard substances (and, hence, absolute densities when the absolute density of the standard is known), are explained in books on Hydrostatics and Physics. The *specific gravity* of a body (which is commonly defined as the ratio of the *weights* of equal volumes) is equivalent to its relative density.

Obs. 2.—When the volume and density of a homogeneous body are given its mass is known. Also in the case of a heterogeneous body, when the average density is known, or the law which regulates the change of density from point to point in the body, its mass can be inferred.

The weight of a body in terms of its volume and density (average density when the body is heterogeneous) is given by the formula

$$W = g\rho V.$$

21. RECTILINEAR MOTION.—A free particle moving vertically downwards or vertically upwards is, if we neglect the resistance of the air, acted on only by its weight, which is a uniform force and produces a uniform acceleration of velocity g in the line of motion. The velocity of the centre of inertia of a rigid body is also uniformly accelerated in a vertical direction, Art. 15. Hence the circumstances of the motion of a particle, or of the motion of the centre of inertia of a rigid body, moving vertically downwards or upwards, may be determined by the equations of *Kinematics*, Art. 18, which become, writing g for a,

$$v = V + gt \quad \text{.................. (1)}$$

$$s = Vt + \tfrac{1}{2}gt^2 \quad \text{............... (2)}$$

$$v^2 = V^2 + 2gs \quad \text{.............. (3)}$$

In these equations g is to be taken positive or negative according as the particle is moving downwards or upwards. For when the particle ascends its velocity is retarded by gravity.

When the particle falls from rest at the commencement of the time t we have, as in *Kinematics*, Art. 19, $V = 0$. Hence

$$v = gt \text{..........................} (4)$$
$$s = \tfrac{1}{2}gt^2 \text{......................} (5)$$
$$v^2 = 2gs \text{......................} (6)$$

Also, as in *Kinematics*, Art. 21, the space described in the nth unit of time is

$$V + \tfrac{1}{2}g\,(2n-1);$$

and when the particle falls from rest the space in the nth unit is

$$\tfrac{1}{2}g\,(2n-1).$$

The numerical value of g when a foot and a second are the units of space and time is roughly speaking 32; that is, the velocity acquired by a falling body in one second is 32 feet per second. By equation (5) the space passed over in the first second is 16 feet.

Obs.—The application of the formulæ of this article to the solution of many interesting problems about bodies moving vertically presents no difficulty whatsoever, if the meanings attached to the symbols are carefully remembered. (See *Kinematics*, Art. 18.) The student will find sufficient examples, for practice in the use of the formulæ, given at the end of this chapter.

22. MOTION ON AN INCLINED PLANE.—When a particle slides down a smooth inclined plane, or when its velocity at any instant is along the line of intersection of the inclined plane with a vertical plane perpendicular to it, its velocity will be uniformly accelerated along the line of motion.

For, let θ be the inclination of the plane to the horizon and m the mass of the particle (or the mass of a body when we seek the motion of its centre of inertia). Then, the weight of the particle mg may be resolved into two components $mg \sin \theta$ along the plane and $mg \cos \theta$ perpendicular to the plane. The latter is counteracted by the normal reaction of the plane and does not change the motion along the plane. The component $mg \sin \theta$ along the plane is constant and produces a uniform acceleration of velocity equal to $g \sin \theta$. Hence the circumstances of the motion may be determined from the equations of the last article by writing

$g \sin \theta$ for g, and taking g positive or negative according as the particle moves downwards or upwards.

Obs.—When the plane is rough we must take into consideration the retarding force of Kinetic friction $\mu mg \cos \theta$ and the acceleration of velocity a will be (see also *Statics*, Art. 192)

$$a = g\,(\sin \theta - \mu \cos \theta)$$

when the particle moves down the plane

and $$a = -g\,(\sin \theta + \mu \cos \theta)$$

when the particle moves up the plane.

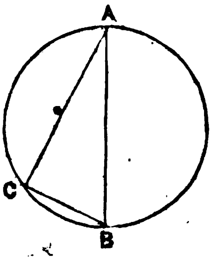
23. *The velocity acquired by a particle in sliding from rest down a smooth inclined plane is the same as would be acquired in falling through a vertical space equal to the height of the plane.*

Let l be the length of an inclined plane whose inclination is θ, and height h; v the velocity acquired in sliding down it from rest. Then by equation (6) of Art. 21, $v^2 = 2gl \sin \theta$. But $h = l \sin \theta$; thus $v^2 = 2gh$. But $2gh$ is equal to the square of the velocity which would be acquired in falling through a vertical space h; which proves the proposition.

Obs.—When a particle slides down any smooth curve the velocity acquired from rest is the same as that which would be acquired in falling freely through a space equal to the *vertical* distance through which the particle moves. This may be proved by considering the curve as made up of an indefinite number of inclined planes. We shall, however, show, in another chapter, that it is a consequence of the Law of Energy.

24. *The time of descent of a particle from rest down any chord of a sphere drawn from the highest point is constant.*

Let AB be the vertical diameter of the sphere, AC any chord drawn through A; θ the angle CAB which is the complement of the inclination of AC to the horizon. The diagram represents the vertical circle which is the intersection of the sphere and the plane BAC. Join BC; the angle BCA in a semicircle is a right angle.



Let t be the time of sliding down the chord AC; then by Art. 22

$$AC = \tfrac{1}{2} t^2 g \cos \theta.$$

But $$AC = AB \cos \theta.$$

Therefore $$t^2 = \frac{2AB}{g}.$$

But this value is independent of the inclination of the chord to the vertical, and, moreover, gives the time of falling through a vertical distance equal to the diameter of the sphere; which proves the proposition.

Cor. 1.—Similarly it may be shown that the time of descent, from rest, down any chord CB drawn through the *lowest* point of the sphere is constant and equal to the time of falling down the vertical diameter.

Cor. 2.—As particular cases we have:—the time of descent from rest down any chord of a vertical circle from the *highest* point is constant; the time of descent down any chord of a vertical circle to the *lowest* point is constant.

Obs.—The results of this article lead to the solutions of many curious problems of *lines of quickest descent.* An example will now be given. The student will have no difficulty in constructing a diagram as indicated.

25. *To find the line of quickest descent from a given point to a given line.*

Let A be the given point which is supposed to lie in a vertical plane passing through the given line. Through A draw a horizontal line to meet the given line in B, and describe a circle having A for its highest point and touching AB at A, and also touching the given line. (The centre of this circle is evidently the point of intersection of the vertical line through A and the line bisecting the angle between AB and the given line, hence the circle may be described.) Let C be the point of contact of the circle and the given line; then AC is the line of quickest descent.

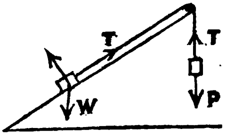
For, draw any other chord AD of the circle, through A, and produce it to meet the given line in E.

Then, the time down AD is equal to the time down AC; but the time down AD is less than the time down AE.

Therefore the time down AC is less than the time down any other line AE which can be drawn from the given point to the given line.

Cor.—Similarly we may find the line of quickest descent from a given line to a given point. The construction is almost the same, but the circle must, in this case, be described so as to have the given point for its *lowest* point. In both cases the distance of the required point C from B is equal to BA, but in the first case C is below B, and in the second case above B, on the given line.

26. *A body, whose weight is* W, *placed on a smooth inclined plane is connected by a string passing over a smooth rail at the top of the plane with another body, whose weight is* P, *hanging freely; to determine the circumstances of motion.*



In this case we may evidently consider the bodies as particles, as far as their motion is concerned. The string between the rail and the body on the plane is supposed to be parallel to the plane, and the weight of the string is neglected. Let us suppose that the body on the inclined plane is ascending while the other body descends. Since the string is supposed to be stretched during the motion, and also to be inextensible, the amount of acceleration of velocity of each body is the same. Also, by the third law of motion, the tension of the string is the same throughout its length.

Let a be the amount of acceleration of velocity and T the tension of the string at any instant; also let θ be the inclination of the plane to the horizon.

The force which produces the acceleration of velocity a in the body whose mass is $\frac{W}{g}$ is the force $T - W \sin\theta$ parallel to the plane. Hence, Art. 8,

$$T - W \sin\theta = \frac{W}{g} a.$$

Similarly $$P - T = \frac{P}{g} a.$$

Solving for the unknown quantities T and a from these equations we get

$$T = \frac{PW}{P+W}(1 + \sin\theta).$$

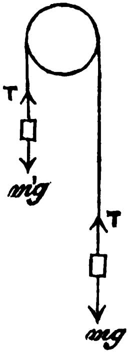
$$a = \frac{P - W\sin\theta}{P+W}g.$$

The first of these equations gives the tension of the string and shows that it is constant during the motion. The second equation gives the amount of acceleration of velocity and shows that it is uniform. The velocity acquired and the space passed over in any given time may be found, when the initial velocity is given, from the equations, *Kinematics*, Art. 18.

Cor. 1.—The condition of uniform motion (or of equilibrium) is found by putting $a = 0$ to be $P = W\sin\theta$. (See also *Statics*, Arts. 172 and 190.)

Cor. 2.—When $P - W\sin\theta$ is negative the acceleration of velocity a is negative and the motion which we have supposed (when the body on the plane moves up) would be retarded.

27. *Two bodies, whose masses are* m *and* m′, *are connected by a string passing over a smooth fixed pulley; to determine the circumstances of motion.*



Let us suppose that the body whose mass is m descends, while the other body ascends.

The weights of the bodies are mg and $m'g$ and the weight of the string is neglected. Let T be the tension of the string, which by the third law of motion is the same throughout its length; and a the amount of acceleration of velocity, which is the same for both bodies, since the string continues stretched and is inextensible.

The vertical force $mg - T$ produces an acceleration of velocity a in a body whose mass is m. Hence, *Art.* 8,

$$mg - T = ma.$$

Similarly $$T - m'g = m'a.$$

Solving between these equations for the unknown quantities T and a, we get

$$T = \frac{2mm'}{m + m'} g.$$

$$a = \frac{m - m'}{m + m'} g.$$

The first equation gives the tension of the string. The second equation gives the acceleration of velocity and shows that it is uniform. The velocity acquired in a given time or in passing over a given space, and the space passed over in a given time, may now be found, when the initial velocity is given, by the equations, *Kinematics*, Art. 18.

Cor. 1.—The condition of uniform motion (or of equilibrium) is found by putting $a = 0$ to be $m = m'$.

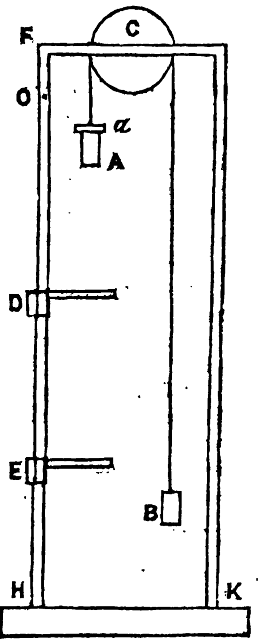
Cor. 2.—When $m - m'$ is negative the motion which we have supposed to take place would be retarded.

Obs. 1.—Nothing has been said as to the motion of the pulley, and thus the results obtained are only approximately true. When the mass of the pulley is small the error thus introduced is slight. Theoretically we might suppose the string to pass round a smooth rail instead of a pulley, but practically, owing to friction, a pulley is a better arrangement. (See *Statics*, Art. 163.)

Obs. 2.—The system of two bodies whose motion is here described forms the essential part of a machine invented by Atwood for testing experimentally results derived from the laws of motion and of falling bodies, and for determining the numerical value of g. The velocity produced in a body, falling freely, by its weight is too great to be convenient for direct observation. By Atwood's machine the motion without being altered in kind is rendered slow enough to be observed directly. In fact by diminishing the difference between m and m' we may make the acceleration of velocity as small as we please.

28. **Atwood's Machine.** — *A* and *B* are bodies of equal masses connected by a string passing over a pulley *C*, which is supported by a vertical frame *FK*. The pulley is supposed to move without friction and its inertia (resistance to motion), and the weight of the string will be neglected.

A body *a* can be placed over *A*, and by this means motion can be communicated to the system. *D* is a moveable platform containing a hole, which allows the body *A* to pass, but which intercepts the body *a*. When the body *a* is intercepted by the platform *D*, the forces acting on the body *A* equilibrate, and it will move on with the velocity which it has acquired until the motion is stopped by a second moveable platform *E*. There is a scale on *FH* by which the spaces passed over can be measured. The times of moving over these spaces are also to be observed by means of a clock in connection with the machine.



Let *M* be the mass of *A*, which is equal to that of *B*, and *m* the mass of *a*. Then, assuming the laws of motion and the laws of falling bodies, a force mg produces an acceleration a, suppose, in a mass $2M + m$. Hence, Art. 26,

$$mg = (2M + m)\,a;$$

or

$$a = \frac{m}{2M + m}\,g.$$

Thus the acceleration of velocity produced is less than the acceleration of velocity of a falling body in the ratio of $m : 2M + m$, and the motion may be made as slow as we please.

To determine a by experiment, the time t of accelerated motion from rest, at any point O, to D, and the time t of uniform motion from D to E, when the body a has been removed, are observed. Let v be the velocity generated and let $DE = s$. Then $s = vt'$ and $v = at$; from which a is found.

Hence, assuming the value of g, we may compare results derived from calculation, from the above equation, with results of observation. Or, if we assume the truth of the equation, the numerical value of g may be calculated from the observed value of a.

Obs.—In practice certain contrivances called *friction wheels* are used for diminishing the friction of the pulley on its axis, and when friction has been diminished as much as possible, the retardation remaining from this cause can be allowed for. The inertia of rotation of the pulley can also be allowed for, in making experiments. Our limits do not however allow us to dwell on these refinements. A more satisfactory method for determining the value of g will presently be described.

29. EXPERIMENTS WITH ATWOOD'S MACHINE.—Various experiments may be made with Atwood's machine with the object of verifying particular cases of the laws of motion and the laws of falling bodies. Some of these will now be shortly noticed.

(1). To verify the first law of motion, assuming that the weight of a body is a constant force.

By making experiments with the platform E in different positions we can find different corresponding values for t' and s (Art. 27), and it will be always found that s is proportional to t', which shows that the velocity continues uniform, when the body a is removed. It is assumed that the weights of the bodies A and B continue constant during the motion, and in equilibrium with each other by means of the connecting string.

(2). To verify that the acceleration of velocity produced is directly proportional to the moving force and inversely proportional to the mass moved, assuming that the weight of a body is proportional to its mass. (Without this assumption there would be no means of estimating the masses of the bodies, which we experiment upon, numerically.)

By using different masses M and m, the sum of the masses $2M + m$ may be kept constantly the same, while the mass m varies. It will be found that a is always directly proportional to m.

Again, keep m constant and vary M, and hence $2M + m$. It will be found that a is always inversely proportional to $2M + m$.

Hence, when the mass moved is constant, the acceleration of velocity produced is directly proportional to the moving force (the weight of the body a, which is proportional to its mass m), and when the force is constant, the acceleration is inversely proportional to the mass moved. Therefore, when both vary, the acceleration of velocity is directly proportional to the moving force and inversely proportional to the mass moved, which is equivalent to a portion of Newton's second law.

(3). Assuming the laws of motion, to verify that the weight of a body is proportional to its mass.

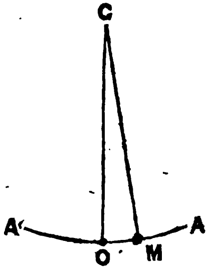
If experiments be made with the platform D in different positions, it will be found that the acceleration of velocity is uniform. But this acceleration of velocity is produced in a given mass by the weight of the body a. Hence the weight of a body is a uniform or constant force.

Again, let p be the weight of a, and P the weight of A, or of B. The force p produces an acceleration a in a given mass; let g be the acceleration which would be produced in the same mass by the force $2P + p$; that is, the acceleration which would be produced in the mass by its own weight. Then by the second law of motion.

$$\frac{g}{a} = \frac{2P + p}{p}.$$

In all cases a can be determined by Atwood's machine, and whatever bodies be experimented upon the same value will be found for g. This verifies (Art. 9) that the weight of a body is proportional to its mass.

80. **The Simple Pendulum.**—When a heavy particle M is suspended by a light inextensible thread from a fixed point C, its position of equilibrium is at O, vertically below C. If the particle be drawn aside from this position and then allowed to fall, it will oscillate about O.



Let θ be the angle MCO which the string makes with the vertical, in any position of the particle. The weight mg of the particle may be resolved into two forces; one $mg \cos\theta$ along CM, is counteracted by the tension of the string; the other $mg \sin\theta$, along the tangent at M, produces the motion.

When the displacement is small (not more than about 3° or 4° from the vertical) the arc OA may be considered to coincide with its chord, and $\sin\theta = \theta$ (in circular measure). Hence the force which produces motion is $mg\theta$ and the acceleration of velocity is $g\theta$, directed to the fixed point O.

Let l be the length of the string, a the acceleration of velocity, and let $OM = x$. Then $x = l\theta$. Hence, since the acceleration of velocity is towards O,

$$a = -g\theta = -\frac{g}{l}x.$$

The acceleration of velocity is thus proportional to the distance of the particle from a fixed point on its path, and is always directed towards the fixed point. The motion is therefore a *simple harmonic* motion (*Kinematics*, Art. 41) whose period is

$$2\pi\sqrt{\frac{l}{g}}$$

The time t of an oscillation of the pendulum from side to side is

$$t = \pi\sqrt{\frac{l}{g}}.$$

The time of oscillation of a pendulum is thus independent of the length of the arc of vibration when the arc is small.

The small vibrations of a pendulum are, hence, said to be *isochronous.*

Cor. 1.—If L be the length of a *seconds pendulum,* that is, of a pendulum beating seconds, we have $t = 1$ and $g = \pi^2 L$. This furnishes the most accurate method of determining the values of g at different parts of the earth's surface. For example, the length of the seconds pendulum in London is 39·14 inches, from which, $g = 32·1908$.

Cor. 2.—The time of oscillation of a pendulum is directly proportional to the square root of its length and inversely proportional to the square root of the apparent force of gravity on the unit of mass. It was shown by Newton that in the same place pendulums of the same length vibrate in equal times whatever be the material of which they are formed. Hence, g is constant for all bodies, and thus the weight of a body is proportional to its mass.

Obs.—In the simple pendulum the string is supposed to have no weight, and the *bob* is supposed to be a particle. In making experiments to verify the laws of a simple pendulum these conditions should be as nearly as possible satisfied. Thus the bob should be formed of a small sphere of some dense material, and the thread should be as fine and light as possible. Such a pendulum oscillates almost like a simple pendulum, whose length is equal to the distance of the centre of the sphere from the point of suspension. A *compound pendulum* is a rigid mass oscillating round a horizontal axis. It is shown in treatises on the Kinetics of a rigid body that the time of a small oscillation of such a body is the same as the time of oscillation of a simple pendulum whose length is the distance between the axis of suspension and a certain line in the body, which is called the *axis of oscillation.*

31. PROJECTILES.—A particle projected in any manner from the surface of the earth is, if we neglect the disturbing influence of the air, acted on only by its weight, which is a constant vertical force, and produces a uniform acceleration of velocity g in a vertical direction downwards. When a particle is projected, in any manner, in any direction, which is not vertical, the circumstances of its motion may be deter-

mined as in *Kinematics*, Art. 43. It is there shown how the position and velocity of the particle can be determined at any instant when the initial velocity is given in magnitude and direction; and that the path described, which is called the *trajectory*, is a parabola. The same is true of the motion of the centre of inertia of a rigid body similarly projected (Art. 15). Some additional solutions of problems about projectiles will now be given.

32. *A particle is projected in a given direction, not vertical, with a given velocity; to determine its motion.*

Let O be the point from which the particle is projected. Through O draw Oy a vertical line, and Ox a horizontal line in a vertical plane, passing through the direction of projection. Let V be the velocity of projection and θ the angle which it makes with Ox. We shall estimate the velocity and acceleration of velocity of the particle by their resolved components parallel to Ox and Oy. Let x and y be the co-ordinates of the particle, v the resultant velocity, and ϕ the angle which its direction makes with Ox, at the end of a time t.

Then, since the velocity parallel to Ox is constant, and the velocity parallel to Oy uniformly retarded by an amount g, we have

$$v \cos \phi = V \cos \theta \qquad (1)$$

$$v \sin \phi = V \sin \theta - gt \qquad (2)$$

$$x = Vt \cos 0 \qquad (3)$$

$$y = Vt \sin \theta - \tfrac{1}{2} gt^2 \qquad (4)$$

The magnitude of the velocity may be found at any given time by squaring the equations (1) and (2) and adding; its direction by dividing equation (2) by equation (1). The position of the particle at any time is determined by equations (3) and (4). Thus the circumstances of the motion are completely determined.

Cor. 1.—The *time of flight* on a horizontal plane is the interval between the particle's projection and its return to the horizontal plane. It is found from equation (4), by putting $y = 0$, to be

$$t = \frac{2V \sin \theta}{g}.$$

Cor. 2.—The *range* on a horizontal plane is the distance from the point of projection to the point where the particle meets the horizontal plane. It is found, by substituting in equation (3) the value of the time of flight, to be

$$x = \frac{2V^2}{g} \sin\ \theta \cos \theta = \frac{V^2}{g} \sin 2\theta.$$

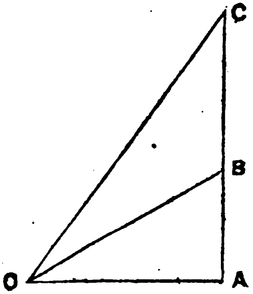
Hence for a given velocity the range is greatest when the angle of projection is 45°.

Cor. 3.—By eliminating t from equations (3) and (4) we get

$$y = x \tan \theta - \frac{gx^2}{2V^2 \cos^2\theta}.$$

This relation between the co-ordinates x and y of the particle must be true for every point on the path. It is the equation of a parabola passing through the origin O and having its axis vertical. (See also *Kinematics*, Art. 43.)

33. *To find the time of flight and range of a projectile on an inclined plane passing through the point of projection and at right angles to the plane of motion.*



Let OB be the intersection of the inclined plane, and OA the intersection of the horizontal plane, with the plane of motion; let OC be the direction of projection; angle $COA = \theta$; angle $BOA = \beta$.

Let B be the point where the projectile meets the inclined plane, and t the time of flight. Through B draw the vertical line ABC. We shall estimate the velocity and acceleration of velocity by their components parallel to OC and a vertical line. Thus

$$OC = Vt;\ CB = \tfrac{1}{2}gt^2.$$

But
$$\frac{CB}{CO} = \frac{\sin(\theta - \beta)}{\cos \beta}.$$

Therefore $$t = \frac{2V \sin(\theta - \beta)}{g \cos \beta}.$$

Again $$\frac{OB}{OC} = \frac{\cos \theta}{\cos \beta}.$$

Therefore $$OB = \frac{2V^2 \sin(\theta - \beta) \cos \theta}{g \cos^2 \beta}.$$

Otherwise.—Let the velocity and acceleration of velocity be estimated by their resolved parts parallel and perpendicular to the plane OB, and let x and y be the co-ordinates of the projectile parallel and perpendicular, respectively, to the plane. Then

$$y = Vt \sin(\theta - \beta) - \tfrac{1}{2} t^2 g \cos \beta.$$

The projectile reaches the plane when $y = 0$. Thus the time of flight is

$$t = \frac{2V \sin(\theta - \beta)}{g \cos \beta}.$$

Again, substituting this value of t in the value of x, the range is found to be

$$x = Vt \cos(\theta - \beta) - \tfrac{1}{2} t^2 g \sin \beta$$
$$= \frac{2V^2 \sin(\theta - \beta) \cos \theta}{g \cos^2 \beta}.$$

Cor.—When the particle is projected down the plane, instead of up, it will be found that the range and time of flight may be got from the results here obtained by changing the sign of β. The range and time of flight on a horizontal plane may also be found as in this article. It will be also seen that the results may be got by putting $\beta = 0$.

Obs.—The theory of projectiles here given is of no practical value as we have neglected the force of resistance of the air, which is found, experimentally, to vary approximately as the square of the velocity, and thus exercises a very considerable influence on the motion. Thus the path of a cannon ball is not a parabola, and its range is very much less than what would be given by the results we have arrived at. The motion of a projectile in air is too difficult a problem to be discussed here.

84. **Law of Gravitation.**—*Every particle of matter in the universe attracts every other particle with a force, in the direction of the line joining the two, whose magnitude is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.*

This is Newton's celebrated law of universal attraction. We shall now explain how the laws of falling bodies follow as consequences of this law, and that the weight of a body, which is the force of the earth's attraction on the body, is proportional to the mass of the body.

We shall suppose the earth's figure to be, as is nearly the case, spherical. It is proved in mathematical treatises that, according to the law of gravitation, the attraction of a spherical body on an external particle is the same as if the entire mass of the sphere were condensed into one particle at its centre.

Let M be the mass of the whole earth, r its radius; m the mass of a body, and h its height above the surface. The mutual force of attraction between the earth and the body is, by the law of gravitation,

$$F = k\frac{Mm}{(r+h)^2},$$

where k is the constant of variation, and whose value it is not necessary for our present purpose to specify.

In the case of a body at the earth's surface, whether at rest or falling freely from a small height above it, h is very small in comparison with r, and may be neglected. Thus

$$F = k\frac{Mm}{r^2}.$$

This shows that the weight F of a body, or the force of the earth's attraction on it, is proportional to its mass m.

Again, let a be the acceleration of velocity which the force F produces in the mass m of a falling body, and a' the acceleration of velocity which F produces in the mass M of the earth. Then

$$a = \frac{F}{m} = k\frac{M}{r^2}; \quad a' = \frac{F}{M} = k\frac{m}{r^2}.$$

The acceleration of the relative velocity with which the body approaches the earth, considered at rest, is therefore

$$a + a' = k\frac{M+m}{r^2}.$$

Now the mass m of the body is very small in comparison to the mass M of the whole earth, and may be neglected. Thus the acceleration of velocity of a falling body is

$$k\frac{M}{r^2},$$

and is independent of the mass of the body.

Obs.—We have assumed that the earth is spherical; this is not accurately the case, as was first shown by Newton. The earth's figure is an oblate spheroid formed by the revolution of an ellipse round its axis minor. Thus the value of r, and hence the value of the force of gravity on the unit of mass, varies in different latitudes. We have also neglected the rotation of the earth on its axis; it will presently be shown how this influences the weight of a body.

35. MOTION OF THE MOON ROUND THE EARTH.—It is known from astronomical observations that the moon's orbit round the earth is *approximately* a circle described uniformly in 27·33 days. We can hence calculate the acceleration of the moon's velocity towards the earth. We have, *Kinematics*, Art. 37,

$$a = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2},$$

where R is the radius of the moon's orbit. It is known from observations that $R = 60r$, where r is the earth's radius, and which we may take as 4,000 miles. Substituting for R and T their values in feet and seconds, and reducing we get

$$a = \frac{55\pi^2}{3^4(27{\cdot}33)^2} = \frac{1}{111{\cdot}5}.$$

Now, according to the second law of motion, we conclude that the moon is acted on by a force directed to the earth, and according to the law of gravitation this force is the force of the earth's attraction on the moon. Now if the law of gravitation be true, the acceleration of the moon's velocity ought to be the same as the acceleration which the earth's attraction would produce in a body at the distance of the moon. Let a be this acceleration of velocity, which is also the measure of the force of gravity on the unit of mass at the moon's distance. Then by the law of gravitation

$$\frac{a}{g}=\frac{r^2}{R^2}.$$

Substituting for R its value $60r$, and taking 32·2 as the approximate mean value of g, we have

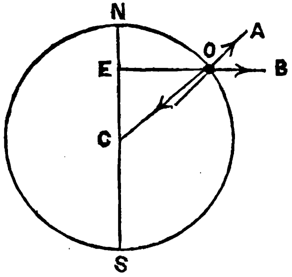
$$a=\frac{32\cdot 2}{(60)^2}=\frac{1}{111\cdot 8}.$$

This value, thus obtained from theoretical considerations, agrees very approximately with the above value calculated from the results of observations; and we have thus a remarkable verification of the truth of the laws of motion and gravitation.

Obs.—The comparison given in this article is celebrated as being that by which Newton first tested his law of gravitation. In the year 1665 Newton formed the opinion that the same force which causes bodies to fall at the earth's surface extended to the moon and retained her in her orbit round the earth. On first making the caculation Newton found that the results did not agree. This was owing to his using the, then, commonly received value of the earth's radius, which was incorrect. Newton laid aside his calculations until 1684. About that time a more correct value of the earth's radius had been calculated, which he then applied to the same problem and obtained the desired agreement.

36. *To determine the effect produced by the earth's rotation on the weight of a body, in any latitude, supposing the earth to be a sphere.*

Let O be the position of a body, whose mass is m, on the earth's surface; NS the earth's axis of rotation. Draw OE



perpendicular to *NS*. Let *OA* be the direction of the pressure of the earth on the body, which is equal and opposite to the weight of the body. *OA* very nearly coincides with the radius *OC*; let ϕ be the small angle which it makes with it.

In consequence of the earth's diurnal rotation on its axis the body *O* moves in a circle whose centre is *E* and radius *OE*; let ω be the angular velocity of rotation of the earth, and *OE* equal to a.

The forces which maintain circular motion in the body *O* are the force of the earth's attraction mg' in the direction *OC*, g' being the true force of gravity on the unit of mass, and the reaction, equal to mg, of the earth against the body in the direction *OA*, g being the weight of the unit of mass, or the apparent force of gravity on the unit of mass. Hence (Art. 12) the resultant of these forces must be in the direction *OE*, and equal to $ma\omega^2$; and the forces are at every instant in equilibrium with the force of inertia of rotation $ma\omega^2$ in the direction *OB*, Art. 13, (4).

Let l be latitude of the place, which is the complement of the angle *OCE*, and therefore equal to *EOC*; and let r be the earth's radius *CO*. Then $a = r \cos l$. Resolving forces along *OC* we get

$$mg' = mg \cos \phi + ma\omega^2 \cos l.$$

Since ϕ is a very small angle we may take $\cos \phi = 1$, and substituting for a we get

$$g = g' - \omega^2 r \cos^2 l.$$

Thus the weight of the unit of mass is less, in consequence of the earth's rotation, than the true force of gravity on the unit of mass by a quantity which is proportional to the square of the cosine of the latitude. The diminution of weight is therefore greatest at the equator, and is zero at the poles.

Cor.—Let a be the force of inertia caused by rotation (centrifugal force) on the unit of mass at the equator where $l = 0$, we have

$$g = g' - a$$

where $$a = \omega^2 r = \frac{4\pi^2 r}{T^2}.$$

The numerical value of a may be found by substituting for r the number of feet in the earth's radius and for T the number of seconds in the sidereal day. It will be sufficiently approximate, however, for our present purpose, to take $r = 4{,}000$ miles and $T = 24$ hours. We thus get $a = \cdot 1116$, and since g at the equator is $32\cdot 088$, we get, very nearly,

$$a = \cdot 1116 = \frac{g}{288}$$

Hence $$288a = g' - a$$

Or $$a = \frac{g'}{289} = \frac{g'}{(17)^2}.$$

If the earth were to revolve 17 times as fast as it actually does revolve, a would be increased in the ratio of 1 to 17^2, and would, hence, become about equal to g'. Thus bodies would have no weight at the equator.

Obs.—We have supposed the earth's figure to be spherical. This is not accurately the case, its figure is an oblate spheroid, and is such that the direction of the weight of a body, which is the resultant of true gravity and the force of inertia caused by rotation, is everywhere perpendicular to its surface. We have shown that the loss of weight caused by rotation is greatest at the equator and diminishes from the equator to the poles. The force of the earth's attraction increases from the equator, where it is least, to the poles, where it is greatest, because the force is inversely as the square of the radius of the earth and the radii (the figure not being spherical) diminish from the equator to the poles. Hence, for both reasons, the weight of a body is least at the equator and increases to the poles, where it is greatest. (See *Statics*, Art. 10.)

EXAMPLES.

1. Find the spaces described by a body falling from rest in the first, second and third seconds, respectively, of its descent. ($g = 32$.)

Ans. 16 ft.; 48 ft.; 80 ft.

2. Find the space described in nine seconds by a body falling from rest.

Ans. 1,296 ft.

3. Find the space described in the ninth second by a body falling from rest.

Ans. 272 ft.

4. Prove the following approximate rule:—"The height of any place in feet is equal to the square of the number of quarter seconds occupied by a body in falling from the top to the bottom."

5. Find the velocity acquired by a body in falling through 225 feet vertically from rest.

Ans. 120.

6. Find the height to which a body, projected vertically upwards with a velocity of 80 feet per second, will ascend, taking $g = 32$.

Ans. 100 ft.

7. A stone is projected vertically upwards with a velocity of 150 feet per second; one second after another stone is projected with a velocity of 200 feet per second; *when* and *where* will the stones meet?

Ans. 2·634 secs.; 284 ft.

8. With what velocity must a stone be projected from the top of a tower 250 yards above the sea, that it may reach the water in 6 seconds?

Ans. 28·4.

9. A stone is projected vertically downwards with a velocity of 50 feet per second; if a second stone be projected after it in 10 seconds, when will it overtake the first stone, the velocity of projection of the second stone being eight times that of the first?

Ans. 80 secs.

10. A person drops a stone into a well, and after t seconds hears it strike the water; assuming the velocity of sound to be $34g$, show that the depth x of the well may be found from the equation

$$x + 34\sqrt{2gx} = 34gt.$$

11. A stone falls from the mouth of the shaft of a mine, and 3 seconds after another stone falls from a point 600 feet lower down. When and where will the first stone overtake the second?

Ans. 961 ft.; 7·75 secs.

12. A body is dropped from a given height, and at the same instant another is projected upwards, and they meet half way; find the initial velocity of the latter body.

Ans. $\sqrt{gh}$.

13. A falling body describes one-third of the entire height in the last second; find the height and the time of descent.

Ans. 5·449 secs.; 475 ft.

14. A falling body describes the n^{th} part of the height in the last second; find the time of falling.

Ans. $n \pm \sqrt{n(n-1)}$.

15. A stone is dropped from a height of 400 feet, and when it has fallen through 50 feet, a second stone is projected after it, so that both stones reach the ground together; find the velocity of projection.

Ans. 72.

16. A body is projected vertically upwards, and the time, n seconds, between its leaving a given point, whose height is h, and returning to it again, is observed. Find the velocity of projection.

Ans. $V^2 = \frac{n^2g^2}{4} + 2gh$.

17. With what velocity must a body be projected downwards, so as in n seconds to overtake another body which has already fallen through a distance h?

Ans. $V = \frac{h}{n} + \sqrt{2gh}$,

18. It is said that at one of the asteroids, a man, who on the earth could leap a height of 6 feet, could jump 60 feet high; compare the force of attraction of the asteroid at its surface with the force of the earth's attraction at its surface.

Ans. $\frac{1}{10}$.

19. Taking the value of g to be 32 when the units of time and space are one second and one foot respectively, find the measure of g when the unit of space is a mètre (39·37 inches).

Ans. 9·8.

20. Taking 32 as the measure of the acceleration of velocity produced by gravity, when the units of length and time are

one foot and one second, find its measure when the units are one yard and one minute.

Ans. 39,600.

21. Find the unit of length when the measure of acceleration of velocity produced by gravity is 14, and the unit of time 5 seconds.

Ans. 55·5.

22. If the weight of the unit of mass be taken as the unit of force, and a foot as the unit of space, what would be the unit of time?

Ans. $\frac{1}{\sqrt{32}}$ secs.

23. The time occupied by a body in running down a smooth inclined plane is n times the time occupied by another body in falling down its height; find the inclination θ of the plane to the horizon.

Ans. $\sin\theta = \frac{1}{n}$.

24. A smooth plane is inclined at an angle of 30° to the horizon; a body is projected up the plane with a velocity g; find the time it takes to describe a space g.

Ans. 2 secs.

25. Divide an inclined plane whose length is l into n parts, such that a body running down the plane shall describe the parts in equal times.

Ans. $\frac{l}{n^2}$, $\frac{3l}{n^2}$, $\frac{5l}{n^2}$, &c.

26. Find the velocity with which a body should be projected down a smooth inclined plane, whose length is l and height h, so that the time of running down the plane shall be equal to the time of falling from rest down the height.

Ans. $V = \frac{g}{l} \cdot \frac{l^2 - h^2}{\sqrt{2gh}}$.

27. Find the inclination of a plane, when a velocity of $\frac{3}{4}$ths that due to the height is sufficient to render the times of running down the plane, and of falling down the height, equal to each other.

Ans. 30°.

28. Determine the inclination to the vertical, of that radius of a vertical circle down which a body will fall in the same time as down the vertical diameter.

Ans. 60°.

29. Two circles lie in the same vertical plane, the lowest point of one being in contact with the highest point of the other; prove that the time of descent from any point of the former to a point in the latter along a straight line joining these points and passing through the point of contact, is constant.

30. If two circles touch each other at their highest or lowest point, and a straight line be drawn through this point, the time of falling from rest down a straight line intercepted between the circumferences is constant.

31. Find the line of quickest descent; (1) from a given point to a given plane; (2) from a given plane to a given point.

32. Find the line of quickest descent:—

(1) From a given point without a given circle to the circle.
(2) From a given circle to a given point without it.
(3) From a given point within a given circle to the circle.
(4) From a given circle to a given point within it.
(5) From a given circle to a given straight line without it.
(6) From a given straight line without a given circle to the circle.
(7) From a given circle to another given circle without it.
(8) From a given circle to another given circle within it.

33. *A* is the highest point of a vertical circle, and *AB* any chord: a circle is described on *AB* as diameter, and a tangent, drawn at *B* to the former circle, meets the latter in *C*; prove that the time of descent down *CB* is constant.

34. Any number of smooth fixed straight rods, not in the same plane, pass through a given point; and a heavy particle slides down each rod, the particles starting simultaneously from the given point. If the rods be so situated that the particles are at one instant in the same plane, prove that they will always be in one plane, and that a circle can be described passing through them.

35. If a body whose mass is 1,000 lbs. be constrained to move in a circle, whose radius is 100 feet, by means of a string, capable of supporting the weight of a mass of 450 lbs., calculate at what velocity the string will break.

Ans. 38·06.

36. If a railway carriage, whose mass is 7·21 tons, moving at the rate of 30 miles per hour, describe a portion of a circle, whose radius is 460 yards, calculate its centrifugal force in gravitation units.

Ans. 0·314 tons.

37. Two bodies whose masses are m and m' are placed on two smooth inclined planes, having a common height, and whose inclinations are θ and θ'. The bodies are connected by a string passing over a smooth small pulley fixed at the top of the planes; required to find the acceleration of velocity, and the tension of the string.

Ans. $\dfrac{m \sin \theta - m' \sin \theta'}{m + m'} g$;

$\dfrac{m m' g}{m + m'} (\sin \theta + \sin \theta')$.

38. A body whose mass is m, hanging vertically, is connected by a string with another body of mass m' placed on a smooth horizontal table, the string passing over a small smooth pulley at the edge of the table; find the acceleration of velocity and the tension of the string.

Ans. $\dfrac{m}{m + m'} g$;

$\dfrac{m m'}{m + m'} g$.

39. A descending body, whose mass is m, draws another body whose mass is m' up an inclined plane whose height and length are h and l, by a cord passing over a small pulley at the summit of the plane; find the distance of m' from the bottom of the plane, when the cord should be cut in order that m' may just reach the top of the plane.

Ans. $\dfrac{m + m'}{m} \cdot \dfrac{h\, l}{h + l}$.

40. In example (38) suppose the table to be rough and μ to be the coefficient of kinetic friction.

Ans. $\dfrac{m - \mu m'}{m + m'} g$;

$\dfrac{m m' g}{m + m'} (1 + \mu)$.

41. A mass of 5 ounces draws a mass of 4 ounces over a smooth pulley; required the space described in 3 seconds from rest.

Ans. $\frac{1}{2} g$ ft.

42. A mass of 7lbs. draws a mass of 5lbs. over a fixed pulley; at the end of the first second the string is cut; required the velocity of the mass of 7lbs. at the end of the next second.

Ans. $\frac{7}{6} g$.

43. A mass of 7lbs. is drawn along a smooth horizontal table by a mass of 1lb. hanging vertically; required the space described in 4 seconds.

Ans. g ft.

44. If a mass of 10lbs. be placed upon a horizontal plane which is made to descend with uniform acceleration of velocity measured by 10; find the mutual pressure between the body and the plane.

Ans. $10(g-10)$.

45. In what time will a force, equal to $5g$, move a mass of 10lbs. through 50 feet along a smooth horizontal plane, and what will be the velocity acquired?

Ans. $\sqrt{\frac{200}{g}}$; $\sqrt{50g}$.

46. A body is lifted vertically against gravity through 2 feet in 2 seconds by a constant force equal to $40g$; what is the mass of the body?

Ans. $\frac{40g}{g+1}$ lbs.

47. In Atwood's machine, if in one experiment the two larger masses be each 15 oz. and the smaller 2 oz., and in a second the larger be each 14 oz. and the smaller mass 4 oz., show that the acceleration of velocity in one case is double of that in the other.

48. If the two larger masses used in Atwood's machine be each a pound, and the smaller mass which is lifted off be a quarter of an ounce, show that the velocity generated in four seconds will be approximately that of one foot per second.

49. If n and n' be the number of vibrations made in the same given time by the same pendulum in places where the weights of the unit of mass are g and g', respectively; then, approximately,

$$\frac{g-g'}{g} = 2\,\frac{n-n'}{n}.$$

50. Colonel Sabine observed that the number of vibrations of a pendulum in London, in a mean solar day, was 85945·80. Having conveyed the same pendulum to Paris, he observed the number of vibrations to be 85933·83; calculate the difference of the weights of the unit of mass in London and Paris as a fraction of its total amount.

Ans. $\frac{1}{3590}$.

51. If t and t' be the times of oscillation of two pendulums, at the same place, whose lengths, l and l', differ by a small amount; show that, approximately,

$$t' - t = \frac{t}{2} \frac{l' - l}{l}.$$

52. The adjustment of a pendulum to its proper length may be effected by a nut on the end of the rod, by which the bob can be screwed up or down. If a clock lose two minutes a day, how many turns must be given to the nut to correct its error, supposing the screw to have 50 threads to the inch, and taking the length of the seconds pendulum 39·14 inches?

Ans. 5·4 turns.

53. Prove that the velocity of a projectile at any point on its path is that which would be acquired in falling from the directrix to the point.

54. Find the time in which a projectile reaches its greatest height, and the greatest height.

Ans. $t = \frac{V \sin\theta}{g}$; $h = \frac{V^2 \sin^2\theta}{2g}$.

55. Prove that the same point in a horizontal plane passing through the point of projection may be reached by two projectiles, having the same initial velocity, whose angles of elevation are complemental.

56. Prove that the direction of elevation which corresponds to the maximum range of a projectile on an inclined plane, passing through the point of projection, bisects the angle between the plane and the vertical line passing through the point of projection.

57. Prove that the same point may be reached by two particles, projected with equal velocities in directions equally inclined to the line bisecting the angle between the line joining the given point to the point of projection and the vertical line through the point of projection.

58. If any number of particles be projected from the same point at the same instant, with equal velocities, in different directions; prove that they will all at any future instant be on the surface of a sphere.

59. If two particles be projected from the same point at the same instant, with different velocities, and in different directions, the line joining them will always move parallel to itself.

60. If three particles be projected from the same point, at the same instant, with different velocities, and in different directions, the plane which passes through them will always move parallel to itself.

61. A smooth plane is inclined at an angle of 45° to the horizon and from the foot of it a particle is projected upwards along the plane, with a velocity V, and reaches the top with one-fifth of its original velocity; find at what distance from the foot of the plane it will strike the ground.

Ans. $\frac{16V^2}{25g}$ ft.

62. Find the inclination θ to an inclined plane, at which a particle must be projected in order to strike on the plane at right angles, the inclination of the plane to the horizon being β

Ans. $2 \tan \theta = \cot \beta$.

63. A particle is thrown over a triangle from one end of the horizontal base, and, grazing the vertex, falls upon the other end of the base. If α and β be the base angles and θ the angle of projection, prove that

$$\tan \theta = \tan \alpha + \tan \beta.$$

64. Supposing the mass of the earth to be 49 times the mass of the moon and the radius of the earth 4 times that of the moon, prove that a seconds pendulum carried to the moon would oscillate in $1\frac{3}{4}$ second.

CHAPTER III.

COLLISION OF BODIES.

37. In this chapter we propose to consider in some simple cases the change of motion produced in each of two bodies which come into collision with one another. For simplicity the bodies are supposed to be two spheres having no motion of rotation before or after the shock, and they may, hence, be treated as particles. All bodies are more or less capable of being compressed, and have a tendency to recover their original forms to a greater or less degree, when the compressing force ceases to act. This property is called *elasticity*.

When two spherical bodies whose centres are moving in the same line come into collision the mutual resultant force between them must at every instant during the contact be along this line, on account of the symmetry of the parts of the bodies about the line. The whole duration of the impact may be divided into two parts. During the first part the bodies are compressed; gradually increasing portions of their surfaces come in contact up to the instant of closest approximation, when the mutual forces of resistance to compression have equalized the velocities of the bodies, and they are then moving with the same velocity as one mass. During the subsequent portion of the time of contact the bodies by reason of their elasticity tend more or less to regain their original forms, and during this period the force of restitution arising from elasticity tends to separate them. The motions of two such bodies after impact are calculated from the following principles.

The force, at every instant during the contact, which one of the bodies exerts on the other is, by the third law of motion, equal and opposite to the force which the second body exerts on the first. Hence these forces generate equal changes of momentum in the bodies in opposite directions.

Again, Newton found experimentally, and his conclusions have been verified by subsequent research, that, when no permanent deformation is produced in either body by the impact, the relative velocity of separation after collision is always in a constant ratio to the relative velocity of approach before impact, for the same two bodies. This constant ratio, called the *coefficient of restitution*, is always less than unity, but approaches more nearly to it the harder the bodies are.

Obs. 1.—The whole duration of the contact which takes place in the collision of two bodies is extremely short, probably in ordinary cases not greater than the one thousandth part of a second. Forces which, acting with great intensity for a very short time, generate a finite change of momentum are called *impulsive forces.*

An impulsive force is measured by the whole change of momentum generated by the impulse, and this change of momentum is produced in so short a time that the change of momentum produced during that time by any ordinary force, such as gravity, is altogether inappreciable. For example, in the case of a blow the immediate effect produced on the body which is struck is not sensibly altered by gravity, or any like force which may be acting on the body.

Obs. 2.—The ratio of the velocity of recoil to the velocity of approach, which is here called the coefficient of restitution, will be taken to be constant for the same two bodies for all velocities. It has been proved, however, experimentally that this law is not accurately true for high velocities, for which the coefficient diminishes.

Obs. 3.—What is here called the coefficient of restitution (a name given by Thomson and Tait) is called in most books the *coefficient of elasticity*, and bodies are thus (inaccurately) spoken of as being theoretically *perfectly elastic*, *imperfectly elastic*, or *inelastic* according as the coefficient of elasticity is unity, less than unity, or zero. The student will find these terms frequently used in problems.

Def.—The collision of spheres is said to be *direct* when at the instant of impact their centres are moving along the same line; otherwise the impact is said to be *oblique.*

38. *Two spherical bodies impinge directly on each other; required to find the motion of each after the impact.*

We shall take as a standard case when the bodies are moving in the same direction, in which direction velocities will be considered positive. Let V and V' be the velocities of the bodies before impact; v and v' their velocities after impact, both taken as positive; m and m' the masses of the bodies; and e the coefficient of restitution. V is supposed to be greater than V'.

The momentum lost by one body in collision is $m\ (V-v)$, and the momentum gained by the other is $m'\ (v'-V')$. Hence, by the third law of motion,

$$m\ (V-v)=m'\ (v'-V') \text{..................} \quad (1)$$

Again, the relative velocity of approach is $V-V'$, and the relative velocity of separation (or recoil) is $v'-v$ Hence, by the law of restitution,

$$v'-v=e\ (V-V',) \text{..................} \quad (2)$$

Equations (1) and (2) are sufficient to determine the unknown quantities v and v' in magnitude and in sign. If the signs of these quantities are found to be both positive the bodies continue to move in the same direction after the collision. If the sign of one of them, however, be found to be negative, this would indicate a reversal of the direction of motion.

Cor.—When the bodies are moving in opposite directions before impact the velocity V' of one of them is to be taken negative, in the above equations. The signs of the quantities v and v' are to be considered as unknown, and are to be determined from these equations in the solution of any particular question.

39. The following investigation, leading to the same results as those given in the last article, is instructive.

Let u be the common velocity of the bodies at the instant of closest approximation, Art. 37. The force which acts on either body during the compression is, by Newton's third law, at every instant equal and opposite to the force which acts on the other body. Let R be the total amount of this impulsive action, measured by the whole change of momentum which it produces. Then with the same notation as in the last article,

$$m\,(V-u)=R.$$

$$m'\,(u-V')=R.$$

Again, if we assume that the impulsive action of restitution due to elasticity is e times the impulsive action due to compression, we have

$$m\,(u-v)=eR.$$

$$m'\,(v'-u)=eR.$$

Eliminating R and u from these four equations, the equations of the last article are immediately arrived at.

Cor.—From these equations we find immediately

$$(m+m')\,u=mV+m'V'=mv+m'v'.$$

This also follows, since the bodies at the instant of closest contact must be moving as one mass with a momentum equal to the sum of their momenta before or after impact.

This relation shows, as is otherwise known, Art. 15, that the motion of the centre of inertia of the bodies is not changed by the collision, Art. 3.

40. *A spherical body impinges obliquely in a given direction on a smooth fixed plane; required to determine the amount and direction of the velocity after impact.*

Let the direction of motion before impact make an angle θ with a line perpendicular to the plane; and let the direction of motion after impact make an angle ϕ with the same line.

The impulsive action during contact is, in direction, perpendicular to the smooth plane. Hence, if we resolve velocities parallel and perpendicular to the plane, the velocity parallel to the plane will be unaltered by the collision, and the velocity perpendicular to the plane will be changed according to the law of restitution. Thus,

$$V\sin\theta=v\,\sin\phi.$$

$$eV\cos\theta=-v\cos\phi.$$

From these equations we get

$$v^2 = V^2 (\sin^2\theta + e^2 \cos^2\theta);$$
$$\cot\phi = -e\cot\theta.$$

The negative sign appears as the velocity after impact, perpendicular to the plane, is reversed in direction. Also the directions of velocity before and after impact lie on different sides of the perpendicular to the plane.

Obs.—The angles θ and ϕ are called, respectively, the angles of *incidence* and *reflexion.*

41. *Two smooth spheres, whose centres are moving in the same plane, impinge obliquely; required to determine their motions after impact.*

Let θ and ϕ be the angles which the directions of velocity of one of the bodies, before and after impact, make with the line joining the centres of spheres at the time of collision; θ' and ϕ' the corresponding quantities for the other body.

The impulsive action must, since the spheres are smooth, take place along the line joining their centres. Hence the resolved components of velocity perpendicular to this line will be unaltered by collision, and the resolved motions along the line of centres must obey the laws of direct collision, Art. 38. Thus

$$V\sin\theta = v\sin\phi.$$
$$V'\sin\theta' = v'\sin\phi'.$$
$$m(V\cos\theta - v\cos\phi) = m'(v'\cos\phi' - V'\cos\theta').$$
$$e(V\cos\theta - V'\cos\theta') = v'\cos\phi' - v\cos\phi.$$

These four equations are sufficient to determine, in magnitude and sign, the four unknown quantities, v, v', ϕ, ϕ'.

Obs.—By particular suppositions regarding the values of the coefficient of restitution e and the masses of the bodies, numerous problems, depending on the results arrived at in this chapter, may be proposed for solution. Theoretically we may regard e to have values between unity and zero, including these limits. In most books the two bodies are said to be *perfectly elastic* when $e = 1$, and *inelastic* when $e = 0$. Many curious results are thus obtained. These results will be sufficiently illustrated in the examples appended to this chapter.

EXAMPLES.

1. If two spheres of equal masses come into direct collision; prove, that if the coefficient of restitution be unity they will interchange velocities.

2. A sphere, whose mass is 5lbs., moving with a velocity of 14 feet per second, impinges directly on a sphere whose mass is 3lbs. and moving in the same direction with a velocity of 8 feet per second; find the velocities after impact, supposing $e = \frac{1}{3}$.

Ans. $v = 11$; $v' = 13$.

3. What must be the coefficient of restitution of two balls whose masses are m and m', in order that m impinging directly upon m' at rest, may itself be reduced to rest by the impact?

Ans. $e = \frac{m}{m'}$.

4. A ball, whose mass is 3lbs., moving with a velocity of 14 feet per second, impinges directly on a ball whose mass is 5lbs moving in the opposite direction with a velocity of 8 feet per second; find their velocities after impact, supposing $e = \frac{1}{2}$.

Ans. $v = -6\frac{5}{8}$; $v' = 4\frac{3}{8}$.

5. If a ball impinge on a plane at an angle of 30° with the normal and be reflected at an angle of 60°, find the coefficient of restitution.

Ans. $e = \frac{1}{3}$.

6. Find with what velocity a ball must impinge upon another ball of equal mass moving with a velocity V', that the impinging ball may be reduced to rest after collision.

Ans. $V = -\frac{1+e}{1-e} V'$.

7. If two balls, for which the coefficient of restitution is unity and whose masses are in the ratio of 1 : 3, moving in opposite directions with equal velocities, meet directly, the larger one will remain at rest.

8. A ball falls from a given altitude a upon a hard horizontal plane, and rebounds continually till its whole velocity is lost; find the whole space described, the coefficient of restitution being e.

Ans. $\frac{1+e^2}{1-e^2} a$.

9. A ball slides down a smooth inclined plane, inclination θ, from a given height h; at the bottom of the inclined plane the ball rebounds from a hard horizontal plane: find the range on the horizontal plane.

Ans. $2eh \sin 2\theta$.

10. A particle is projected with a velocity V, at an inclination θ to the horizon, and, by continually rebounding from the horizontal plane, describes a series of parabolas; find the whole range, and the whole time of flight.

Ans. $\dfrac{V^2 \sin 2\theta}{g(1-e)}$; $\dfrac{2V \sin\theta}{g(1-e)}$.

CHAPTER IV.

ENERGY.

42. **Measure of Kinetic Energy.**—A body in motion is capable of performing a definite amount of work against forces which resist the motion. The work which a body can thus perform is called its *kinetic energy*, and we shall now show how it is to be measured. The body will be supposed to have no motion of rotation.

Suppose that a body, whose mass is m, is moving in a straight line with a velocity V at any instant, and that it is acted on by a constant force P in the direction opposite to that of motion. The velocity of the body will be uniformly retarded; let a be the amount of retardation of velocity. Then, Art. 8,

$$P = m\,a.$$

Let v be the velocity of the body when it has moved over a space s against the force P. Then, *Kinematics*, Art. 18,

$$v^2 = V^2 - 2\,a\,s.$$

The body comes to rest when $v = 0$; let h be the corresponding value of s. Then,

$$V^2 = 2\,a\,h.$$

When the body comes to rest all its kinetic energy has been consumed in work done against the force. The work so performed through the space h is Ph, *Statics*, Art. 178. Now, we have,

$$m V^2 = 2h\,m\,a.$$

Hence
$$Ph = \frac{mV^2}{2}.$$

Therefore, *the kinetic energy of a body in motion (without rotation) is half the product of its mass and the square of its velocity.*

Obs.—Let A be the position of the body when its velocity is V, and B its position when its velocity is zero. It is to be observed that the body will not continue at rest at B.

In fact the force P, continuing to act in the same manner, will generate in the body acceleration of velocity directed from B towards A, and when the body returns to A, its velocity, found from the equation $V^2 = 2ah$, is exactly the same as the original velocity. Hence, by work measured by Ph done on the body by the force its original kinetic energy is restored. The body when at B is said to have *potential* energy, with reference to the force P, greater than its potential energy at A by an amount Ph, which is the work which would be done on the body in moving from B to A. Thus the energy of the body at B is equal to its energy at A, but at B its energy is all potential.

Obs. 2.—The product of the mass of a body into the square of its velocity is called the *vis viva* of the body. The *vis viva* is hence twice the kinetic energy.

43. *If the action of an agent be measured by its force and velocity conjointly; and if, similarly, the reaction of the resistance be measured by the velocities of its several parts and their several forces conjointly, whether arising from friction, cohesion, weight or acceleration; action and reaction in all combinations of machines will be equal and opposite.*

Newton gives the above as a scholium to his third law of motion in its application to the Science of Machines, or *Mechanics*, and thus specifies another class of actions and reactions which are subject to his third law.

By the action of a force is here to be understood the rate at which it works, which is proportional to the work done by the force during an indefinitely small displacement of its point of application; that is, to the product of the resolved component of the force in the direction of motion into the displacement of its point of application, *Statics*, Art. 178.

When the point of application of the power, in a machine, moves through any indefinitely small space, the points of application of the several resistances must move, in the same time, through corresponding spaces proportional to their velocities. Hence Newton's statement is equivalent to saying that, during any indefinitely small motion of a machine, the work done by the power is equal to the whole work done against resistances, whether arising from friction, molecular forces, gravity, or acceleration of velocity.

Any finite displacement of the point of application of a force may be supposed to be made up of a great number of very small displacements, and the whole work done by a force is the sum of the several amounts of work for all these displacements. When there is acceleration of velocity part of the work done by the power is expended against reactions to acceleration (that is, against the forces of inertia of the different parts). This work has its equivalent in a corresponding amount of kinetic energy. When work is expended in raising a body against gravity, the fall of the body is capable at any time of restoring the amount of work so spent. When work is expended against molecular forces as in pending a perfect spring, the recoil of the spring is capable of restoring at any time the amount of work so spent. Thus Newton's scholium on his third law stated in modern language is the law of energy as applied to Mechanics :—

The work done, in any time, by the power is equal to the whole effects in the forms of potential and kinetic energy together with the work lost against friction.

The work lost against friction is transformed into *heat*, a form of molecular energy which always accompanies friction.

Obs.—The foundation of the theory of energy in abstract dynamics was thus laid down by Newton in a remarkably clear and compact manner, but the importance and full meaning of this scholium on his third law escaped the notice of commentators until recently pointed out by Thomson and Tait.

44. CONSERVATION OF ENERGY.—The invention of a machine which, without having energy of any kind supplied to it from without, would continue to perform work for ever had long been one of the dreams of enthusiasts. The Principle of the Conservation of Energy denies the possibility of the construction of such a machine by asserting that *the total amount of energy of any body or system of bodies cannot be altered in amount by any mutual actions between the parts of the body or system, although it may*

be transmuted into the different forms of which energy is susceptible. (See, also, *Statics*, Art. 197.)

The conservation of Energy asserts that it is impossible to create or destroy energy. If, however, it were possible to transform and retransform energy by any perfect process, a machine might be constructed which, when once put in motion, would continue to move for ever, but without performing work on external matter. It is known from experience, however, that no process in nature is exactly reversible, so that no system of bodies can pass from one particular configuration through a series of others and back again to the original configuration without having part of its energy transmuted into forms which are incapable of further useful transformation. For example, there is no such thing in nature as a perfectly smooth body, and the motion of a machine is, thus, always retarded by friction, the energy so spent being transformed into diffused heat from which no useful effect can be derived. Such diffused energy is said to be *dissipated,* and the Principle of the Dissipation of Energy asserts that (another of the dreams of enthusiasts) "the production of *perpetual motion,*" of any kind, is impossible.

45. ENERGY IN ABSTRACT DYNAMICS.—The different forms of which energy is susceptible and the transformations from one of these forms into any of the others are treated of in works on Physics and Chemistry. In Abstract Dynamics we are concerned with motion under the action of any forces, but unresisted by friction. We have, therefore, only to consider visible kinetic energy, or the energy which a body possesses in virtue of its motion, and visible potential energy, which a body possesses in virtue of a position of advantage with respect to forces acting on the body, and the transmutations of one of these forms of energy into the other.

The law of energy in Abstract Dynamics is that *the sum of the potential and kinetic energies of any body or system of bodies not acted upon by external forces is constant* (*Statics,* Art. 180), and this can be proved by mathematical reasoning (which is, however, beyond our limits) in the case of a system whose mutual forces depend on the distances of the particles, and not on the way in which the parts are moving at the time. For example, if we take the case of a body in motion under the influence of the earth's attraction, the law asserts that the sum of the kinetic and potential energies of the system, the earth and the body, is constant during the motion. When, however, the body comes to rest by collision with the earth's surface, the apparent loss of energy is to be accounted for by a transformation into energy of vibrations and the invisible molecular energy of heat.

In Abstract Dynamics we also speak of a body or system as being acted upon by external forces, and the law of energy asserts that the work done on the body or system in any time is equal to the whole change of energy. But if the forces are such that they will act equally when any motion is reversed, there will be no change of energy, and the sum of the kinetic energy and the potential energy, with reference to all the forces, internal and external, will be constant. See, also, *Statics*, Art. 180.

46. The energy of a system of bodies whose mutual forces depend on their relative positions is, as has been stated, due partly to their motion and partly to their relative positions. The potential energy, depending on their relative positions only, is the work that the various forces would do during the passage of the body from any one chosen configuration to another, and the law of energy in Abstract Dynamics requires that the same amount of work should be done by whatever set of paths the system passes from the one configuration to the other. For, if more work is done in passing by one set of paths than in passing by any other set, the system might be directed by frictionless constraints to move from the one configuration by one set of paths and return by the other, and so on for ever. We should thus have a continual creation of energy, which is impossible.

Since the sum of the kinetic and potential energies is constant, the kinetic, as well as the potential, energy depends on the particular configuration which the system occupies at any instant, and is the same at all times when the system again returns to that particular configuration.

47. *To verify the law of conservation of energy in the case of a body moving freely under the action of gravity.*

First, let us suppose that the body is in motion along a vertical line; let m be its mass and W its weight, so that $W = mg$; V its velocity at any instant, and v its velocity after moving through a vertical space h. When the body is descending

$$v^2 = V^2 + 2gh.$$

Therefore $$\frac{m}{2}(v^2 - V^2) = Wh.$$

The left hand side of this equation represents the increase of kinetic energy, and the right hand side represents the work done on the body by gravity, which is also the measure of the loss of the body's potential energy with reference to gravity.

When the body is ascending we have

$$v^2 = V^2 - 2gh.$$

Therefore $$\frac{m}{2}(V^2 - v^2) = Wh.$$

The left hand side is now the loss of kinetic energy, and the right hand side is the work done against gravity, which is also the measure of the gain of the body's potential energy with reference to gravity.

Secondly, when the body is moving in any direction not vertical, with the same notation and the equations of Art. 32, we have

$$v^2 = (V \sin\theta - gt)^2 + V^2 \cos^2\theta.$$

$$= V^2 - 2g\left(Vt \sin\theta - \tfrac{1}{2}gt^2\right) = V^2 - 2gy.$$

Therefore $$\frac{m}{2}(V^2 - v^2) = Wy.$$

Since y is the height of the body above the horizontal plane through the point of projection, Wy is the gain of potential energy.

Hence, in all cases of free motion under the action of gravity, the sum of the kinetic and potential energies is constant.

Obs. 1.—During the motion a continual transformation of energy, from kinetic to potential, as the body ascends, and from potential to kinetic, as the body descends, takes place. When a body, projected vertically, is at its greatest height its whole energy is potential, when it returns again to the ground its whole energy is kinetic. We have already mentioned what becomes of this energy after collision with the ground.

Obs. 2.—We have defined the potential energy of a body with reference to gravity to be the work which would be done on the body in falling vertically to the ground, taking the earth's surface as the zero of reckoning for potential energy, *Statics*, Art. 179. When a body slides down an inclined plane it is evident that the work done by gravity is the same as the work done in falling freely through the height, *Statics*, p. 161, Ex. 9; and it is a necessary consequence of the Law of Energy that the work done is the same by whatever path the body may pass from one position to another at a different height above the surface. This is also proved independently in the next article for the case of a particle.

48. *The work done on a particle, moving under the action of gravity, whether freely, or along any smooth curve, is equal to the weight of the particle multiplied by the difference of altitudes between its first and last positions.*

Let A be the first position of the particle and B the last position, and let A and B be joined by any curve whatsoever. Let the curve be supposed to be made up of an indefinite number of elements, each of which may be considered to be an indefinitely small portion of a straight line. Let s be the length of one of these elements and θ the angle which its direction makes with a vertical line.

The weight W of the particle may be resolved into two components $W\cos\theta$ in the direction of motion (or in the opposite direction) and $W\sin\theta$ perpendicular to the direction of motion. Let R be the reaction of the curve on the particle. The forces R and $W\sin\theta$ being perpendicular to the direction of motion do no work. Hence the whole work done on the body by gravity, while the body moves from A to B, is

$$\Sigma sW\cos\theta = W\Sigma s\cos\theta;$$

where the letter Σ is used to denote the sum of similar quantities for the whole path between A and B. But $\Sigma s\cos\theta$ is evidently equal to the difference of altitudes between A and B; which proves the proposition.

Obs.—The motion of the particle may be imagined to take place in a very narrow smooth tube in the form of any curve, or we may suppose a very small ring to move on a very fine wire in the shape of any curve.

49. *The change of the kinetic energy of a particle, moving under the action of gravity along any smooth curve, in passing from any point to any other point, is independent of the form of the curve joining the points.*

This follows from the proposition of the last article and the law of energy, since the sum of the potential and kinetic energies is constant, Art. 46.

Thus let V be the velocity of the particle at one point A, and v its velocity at B; y the vertical distance between A and B. Then, whatever be the form of the path connecting A and B, the work done by, or against, gravity in passing from A to B is mgy, and this is also the difference of potential energies at A and B. Hence,

$$\frac{m}{2}(v^2 - V^2) = mgy.$$

Thus the change of kinetic energy depends only on the mass of the particle and the vertical distance between the points.

Cor.— $$v^2 = V^2 + 2gy.$$

This equation determines the velocity of the particle in any position when descending, V being the initial velocity. When the particle is ascending the negative sign is to be used with g. When the particle descends from rest, we have, $v^2 = 2gy$; see, also, Art. 23. The velocity acquired from rest is therefore that due to a fall through a vertical distance y.

50. *To determine the loss of kinetic energy in the direct collision of two elastic spherical bodies.*

From the equations of Art. 38, we have

$$(m v + m' v')^2 = (m V + m' V')^2$$

and

$$m m' (v' - v)^2 = m m' e^2 (V - V')^2$$

$$= m m' (V - V')^2 - m m' (1 - e^2)(V - V')^2$$

By adding these equations and reducing we get

$$(m + m')(m v^2 + m' v'^2) = (m + m')(m V^2 + m' V'^2) - m m' (1 - e^2)(V - V')^2$$

therefore

$$\tfrac{1}{2}(m V^2 + m' V'^2) - \tfrac{1}{2}(m v^2 + m' v'^2) = \frac{m m'}{2(m + m')}(1 - e^2)(V - V')^2$$

Cor.—If we suppose $e = 1$ we have

$$\tfrac{1}{2}(m V^2 + m' V'^2) = \tfrac{1}{2}(m v^2 + m' v'^2)$$

and thus there would be no loss of kinetic energy caused by the impact.

Obs. 1.—Of the Kinetic energy apparently lost in impact the greater part is transformed into heat. It is to be observed, however, that in Art. 38 we have supposed the momentum lost by one body, moving as a whole, to be equal to the gain of momentum of the other body, moving as a whole. This is never practically the case. Part of the momentum of each body is always changed into the momentum of vibrations of the body itself. Thus part of the kinetic energy apparently lost in impact consists in the energy of the vibrations which are produced, and thus even on the supposition $e = 1$, the Kinetic energy after impact would (if we neglect the energy of vibrations) be less than the energy before impact.

Obs. 2.—The energy of a vibrating body is communicated to any surrounding medium, such as air, and is finally dissipated as diffused heat. This is the form of kinetic energy which, when communicated to our organs of hearing, is called *sound*.

51. ILLUSTRATIONS OF ENERGY AND ITS TRANSFORMATIONS.—In the motions of bodies, (supposed to be unresisted by friction and the resistance of the surrounding medium)

which we consider in Abstract Dynamics, a continual transformation from one of two forms of energy, potential and kinetic, into the other, without any loss from dissipation, would take place. We now give a few common examples of these forms of energy, and transformations from one form into the other. Numerous other examples will readily occur to the reader.

(1). Motive power is frequently supplied to machinery by the fall of water from a higher to a lower level. The loss of potential energy sustained by the water in descending is the energy which is taken up and (in part) transformed into useful work by the machine.

(2). In an air-gun we have a store of potential energy in the compressed air, which is capable of being converted into the kinetic energy of motion of the bullet. The potential energy of a bow when bent is converted into kinetic energy when the arrow is shot from the bow.

(3). The motion of the wheels of a clock is maintained by the transformation of the potential energy of certain masses which descend under the action of gravity. The motion of the wheels of a watch is derived from the potential energy of the watch spring.

(4). The transformations of energy which take place when a body is projected under the action of gravity have already been noticed. See Art. 47.

(5). The motion of a pendulum furnishes a good example of transformations from potential into kinetic energy and *vice versâ*. When the bob is drawn aside from its lowest position to a higher position its energy is all potential. During the swing to the lowest position the potential energy is gradually converted into kinetic energy, and at the lowest position all its energy is kinetic. During the ascent of the bob on the other side, its kinetic energy is gradually reconverted into potential energy, and when the bob reaches the same height from which it fell originally all its energy is again potential. The same series of changes would recur for ever if there were no loss of energy from friction and the resistance of the air.

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